MATHEMATICS

DISCRETE MATH

Symbols

$x \in X$	x is a member of X
$\{\ \}, \phi$	The empty (or null) set
$S \subseteq T$	S is a subset of T
$S \subset T$	S is a proper subset of T
(a,b)	Ordered pair
$P^{(s)}$	Power set of S
$(a_1, a_2, ..., a_n)$	n-tuple
$A \times B$	Cartesian product of A and B
$A \cup B$	Union of A and B
$A \cap B$	Intersection of A and B
$\forall$ x	Universal qualification for all x; for any x; for each x
$\exists$ y	Uniqueness qualification there exists y

A binary relation from A to B is a subset of $A \times B$.

Matrix of Relation

If $A = \{a_1, a_2, ..., a_m\}$ and $B = \{b_1, b_2, ..., b_n\}$ are finite sets containing m and n elements, respectively, then a relation R from A to B can be represented by the $m \times n$ matrix $M_R < [m_{ij}]$, which is defined by:

$m_{ij} = \{$ 1 if $(a_i, b_j) \in R$

0 if $(a_i, b_j) \notin R\}$

Directed Graphs, or Digraphs, of Relation

A directed graph, or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). For edge (a, b), the vertex a is called the initial vertex and vertex b is called the terminal vertex. An edge of form (a, a) is called a loop.

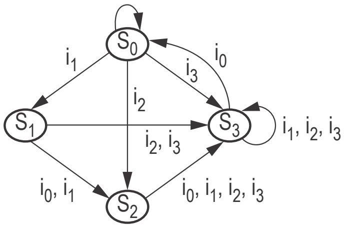
Finite State Machine

A finite state machine consists of a finite set of states $S_i = \{s_0, s_1, ..., s_n\}$ and a finite set of inputs I; and a transition function f that assigns to each state and input pair a new state.

A state (or truth) table can be used to represent the finite state machine.

	Input			
State	i_0	i_1	i_2	i_3
S_0	S_0	S_1	S_2	S_3
S_1	S_2	S_2	S_3	S_3
S_2	S_3	S_3	S_3	S_3
S_3	S_0	S_3	S_3	S_3

Another way to represent a finite state machine is to use a state diagram, which is a directed graph with labeled edges.



The characteristic of how a function maps one set (X) to another set (Y) may be described in terms of being either injective, surjective, or bijective.

An injective (one-to-one) relationship exists if, and only if, $\forall\ x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$

A surjective (onto) relationship exists when $\forall\ y \in Y, \exists\ x \in X$ such that f(x) = y

A bijective relationship is both injective (one-to-one) and surjective (onto).

STRAIGHT LINE

The general form of the equation is

$$Ax + By + C = 0$$

The standard form of the equation is

$$y = mx + b,$$

which is also known as the *slope-intercept* form.

The *point-slope* form is $y - y_1 = m(x - x_1)$

Given two points: slope, $m = (y_2 - y_1)/(x_2 - x_1)$

The angle between lines with slopes m_1 and m_2 is

$$\alpha = \arctan\left[(m_2 - m_1)/(1 + m_2 \cdot m_1)\right]$$

Two lines are perpendicular if $m_1 = -1/m_2$

The distance between two points is

$$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

QUADRATIC EQUATION

$$ax^2 + bx + c = 0$$

$$x = \text{Roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

QUADRIC SURFACE (SPHERE)

The standard form of the equation is

$$(x - h)^2 + (y - k)^2 + (z - m)^2 = r^2$$

with center at (h, k, m).

In a three-dimensional space, the distance between two points is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

LOGARITHMS

The logarithm of x to the Base b is defined by

$\log_b(x) = c$, where $b^c = x$

Special definitions for $b = e$ or $b = 10$ are:

$\ln x$, Base $= e$

$\log x$, Base $= 10$

To change from one Base to another:

$\log_b x = (\log_a x)/(\log_a b)$

e.g., $\ln x = (\log_{10} x)/(\log_{10} e) = 2.302585 (\log_{10} x)$

Identities

$\log_b b^n = n$

$\log x^c = c \log x$; $x^c = \text{antilog}(c \log x)$

$\log xy = \log x + \log y$

$\log_b b = 1$; $\log 1 = 0$

$\log x/y = \log x - \log y$

ALGEBRA OF COMPLEX NUMBERS

Complex numbers may be designated in rectangular form or polar form. In rectangular form, a complex number is written in terms of its real and imaginary components.

$z = a + jb$, where

a = the real component,

b = the imaginary component, and

$j = \sqrt{-1}$ (some disciplines use $i = \sqrt{-1}$)

In polar form $z = c \angle \theta$ where

$c = \sqrt{a^2 + b^2}$,

$\theta = \tan^{-1}(b/a)$,

$a = c \cos \theta$, and

$b = c \sin \theta$.

Complex numbers can be added and subtracted in rectangular form. If

$z_1 = a_1 + jb_1 = c_1(\cos \theta_1 + j\sin \theta_1)$

$= c_1 \angle \theta_1$ and

$z_2 = a_2 + jb_2 = c_2(\cos \theta_2 + j\sin \theta_2)$

$= c_2 \angle \theta_2$, then

$z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2)$ and

$z_1 - z_2 = (a_1 - a_2) + j(b_1 - b_2)$

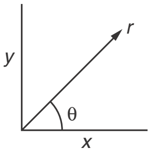
While complex numbers can be multiplied or divided in rectangular form, it is more convenient to perform these operations in polar form.

$z_1 \times z_2 = (c_1 \times c_2) \angle (\theta_1 + \theta_2)$

$z_1/z_2 = (c_1/c_2) \angle (\theta_1 - \theta_2)$

The complex conjugate of a complex number $z_1 = (a_1 + jb_1)$ is defined as $z_1^* = (a_1 - jb_1)$. The product of a complex number and its complex conjugate is $z_1 z_1^* = a_1^2 + b_1^2$.

Polar Coordinate System



$x = r \cos \theta$; $y = r \sin \theta$; $\theta = \arctan(y/x)$

$r = |x + jy| = \sqrt{x^2 + y^2}$

$x + jy = r(\cos \theta + j \sin \theta) = re^{j\theta}$

$[r_1(\cos \theta_1 + j \sin \theta_1)][r_2(\cos \theta_2 + j \sin \theta_2)] =$

$r_1 r_2[\cos(\theta_1 + \theta_2) + j \sin(\theta_1 + \theta_2)]$

$(x + jy)^n = [r(\cos \theta + j \sin \theta)]^n$

$= r^n(\cos n\theta + j \sin n\theta)$

$$\frac{r_1(\cos\theta_1 + j\sin\theta_1)}{r_2(\cos\theta_2 + j\sin\theta_2)} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)\right]$$

Euler's Identity

$e^{j\theta} = \cos \theta + j \sin \theta$

$e^{-j\theta} = \cos \theta - j \sin \theta$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Roots

If k is any positive integer, any complex number (other than zero) has k distinct roots. The k roots of $r(\cos \theta + j \sin \theta)$ can be found by substituting successively $n = 0, 1, 2, ..., (k - 1)$ in the formula

$$w = \sqrt[k]{r}\left[\cos\left(\frac{\theta}{k} + n\frac{360°}{k}\right) + j\sin\left(\frac{\theta}{k} + n\frac{360°}{k}\right)\right]$$

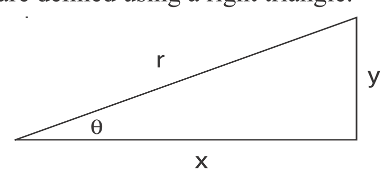
TRIGONOMETRY

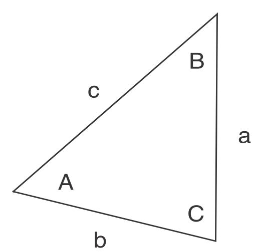
Trigonometric functions are defined using a right triangle.

$\sin \theta = y/r$, $\cos \theta = x/r$

$\tan \theta = y/x$, $\cot \theta = x/y$

$\csc \theta = r/y$, $\sec \theta = r/x$





Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of Cosines

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

Brink, R.W., *A First Year of College Mathematics*, D. Appleton-Century Co., Inc., Englewood Cliffs, NJ, 1937.

Identities

$\cos \theta = \sin (\theta + \pi/2) = -\sin (\theta - \pi/2)$

$\sin \theta = \cos (\theta - \pi/2) = -\cos (\theta + \pi/2)$

$\csc \theta = 1/\sin \theta$

$\sec \theta = 1/\cos \theta$

$\tan \theta = \sin \theta/\cos \theta$

$\cot \theta = 1/\tan \theta$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$

$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

$\sin 2\alpha = 2 \sin \alpha \cos \alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2 \sin^2\alpha = 2 \cos^2\alpha - 1$

$\tan 2\alpha = (2 \tan \alpha)/(1 - \tan^2\alpha)$

$\cot 2\alpha = (\cot^2\alpha - 1)/(2 \cot \alpha)$

$\tan (\alpha + \beta) = (\tan \alpha + \tan \beta)/(1 - \tan \alpha \tan \beta)$

$\cot (\alpha + \beta) = (\cot \alpha \cot \beta - 1)/(\cot \alpha + \cot \beta)$

$\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

$\tan (\alpha - \beta) = (\tan \alpha - \tan \beta)/(1 + \tan \alpha \tan \beta)$

$\cot (\alpha - \beta) = (\cot \alpha \cot \beta + 1)/(\cot \beta - \cot \alpha)$

$\sin (\alpha/2) = \pm\sqrt{(1 - \cos \alpha)/2}$

$\cos (\alpha/2) = \pm\sqrt{(1 + \cos \alpha)/2}$

$\tan (\alpha/2) = \pm\sqrt{(1 - \cos \alpha)/(1 + \cos \alpha)}$

$\cot (\alpha/2) = \pm\sqrt{(1 + \cos \alpha)/(1 - \cos \alpha)}$

$\sin \alpha \sin \beta = (1/2)[\cos (\alpha - \beta) - \cos (\alpha + \beta)]$

$\cos \alpha \cos \beta = (1/2)[\cos (\alpha - \beta) + \cos (\alpha + \beta)]$

$\sin \alpha \cos \beta = (1/2)[\sin (\alpha + \beta) + \sin (\alpha - \beta)]$

$\sin \alpha + \sin \beta = 2 \sin [(1/2)(\alpha + \beta)] \cos [(1/2)(\alpha - \beta)]$

$\sin \alpha - \sin \beta = 2 \cos [(1/2)(\alpha + \beta)] \sin [(1/2)(\alpha - \beta)]$

$\cos \alpha + \cos \beta = 2 \cos [(1/2)(\alpha + \beta)] \cos [(1/2)(\alpha - \beta)]$

$\cos \alpha - \cos \beta = -2 \sin [(1/2)(\alpha + \beta)] \sin [(1/2)(\alpha - \beta)]$

MENSURATION OF AREAS AND VOLUMES

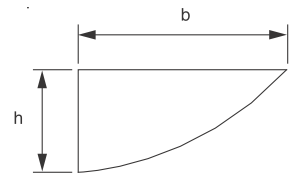
Nomenclature

A = total surface area

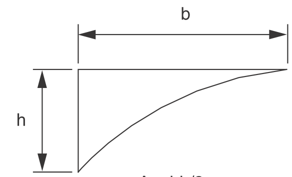
P = perimeter

V = volume

Parabola

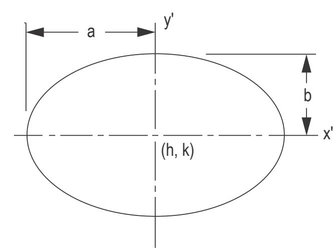


A = 2bh/3



A = bh/3

Ellipse



A = πab

$$P_{approx} = 2\pi\sqrt{\left(a^2 + b^2\right)/2}$$

$$P = \pi\left(a + b\right)\begin{bmatrix} 1 + \left(1/2\right)^2\lambda^2 + \left(1/2 \times 1/4\right)^2\lambda^4 \\ +\left(1/2 \times 1/4 \times 3/6\right)^2\lambda^6 + \left(1/2 \times 1/4 \times 3/6 \times 5/8\right)^2\lambda^8 \\ +\left(1/2 \times 1/4 \times 3/6 \times 5/8 \times 7/10\right)^2\lambda^{10} + \ldots \end{bmatrix},$$

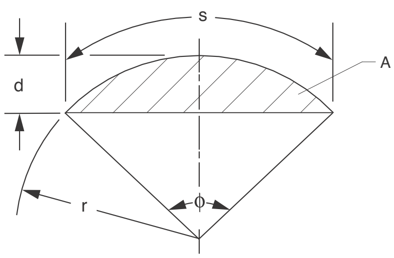
where

$\lambda = \left(a - b\right)/\left(a + b\right)$

Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publishing, 1967.

Circular Segment

♦

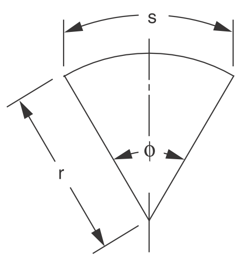


$$A = \left[r^2(\phi - \sin\phi)\right]/2$$

$$\phi = s/r = 2\left\{\arccos\left[(r - d)/r\right]\right\}$$

Circular Sector

♦

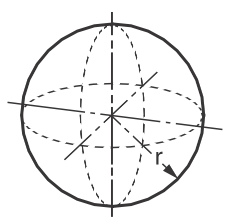


$$A = \phi r^2/2 = sr/2$$

$$\phi = s/r$$

Sphere

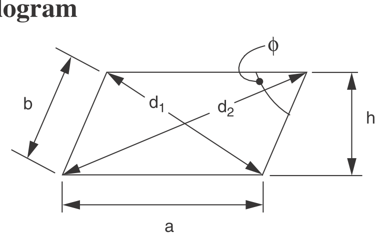
♦



$$V = 4\pi r^3/3 = \pi d^3/6$$

$$A = 4\pi r^2 = \pi d^2$$

Parallelogram



$$P = 2(a + b)$$

$$d_1 = \sqrt{a^2 + b^2 - 2ab(\cos\phi)}$$

$$d_2 = \sqrt{a^2 + b^2 + 2ab(\cos\phi)}$$

$$d_1^2 + d_2^2 = 2\left(a^2 + b^2\right)$$

$$A = ah = ab(\sin\phi)$$

If $a = b$, the parallelogram is a rhombus.

Regular Polygon (n equal sides)

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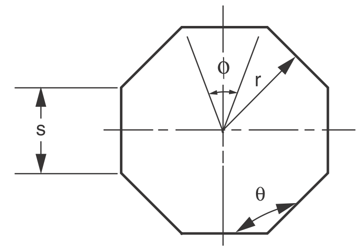


$$\phi = 2\pi/n$$

$$\theta = \left[\frac{\pi(n-2)}{n}\right] = \pi\left(1 - \frac{2}{n}\right)$$

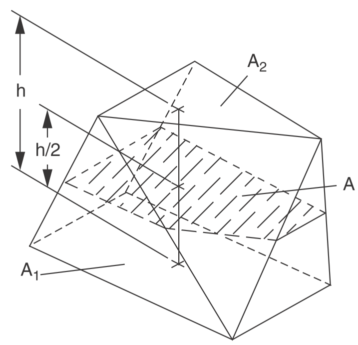
$$P = ns$$

$$s = 2r\left[\tan(\phi/2)\right]$$

$$A = (nsr)/2$$

Prismoid

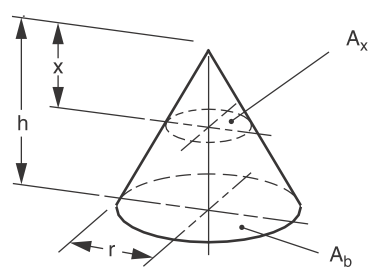
♦



$$V = (h/6)\left(A_1 + A_2 + 4A\right)$$

Right Circular Cone

♦



$$V = \left(\pi r^2 h\right)/3$$

$$A = \text{side area} + \text{base area}$$

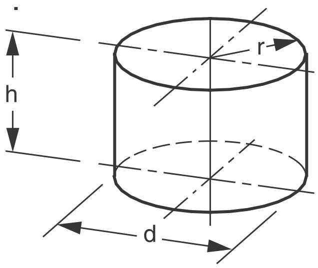
$$= \pi r\left(r + \sqrt{r^2 + h^2}\right)$$

$$A_x : A_b = x^2 : h^2$$

♦ Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publishing, 1967.

Right Circular Cylinder

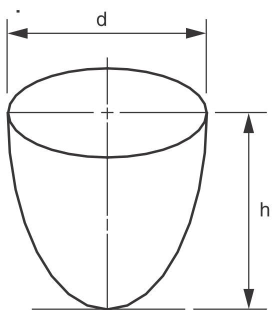
♦



$$V = \pi r^2 h = \frac{\pi d^2 h}{4}$$

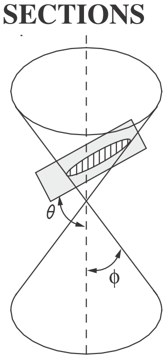
$$A = \text{side area} + \text{end areas} = 2\pi r(h + r)$$

Paraboloid of Revolution



$$V = \frac{\pi d^2 h}{8}$$

CONIC SECTIONS



e = eccentricity = $\cos\theta/(\cos\phi)$

[Note: X' and Y', in the following cases, are translated axes.]

Case 1. Parabola $e = 1$:

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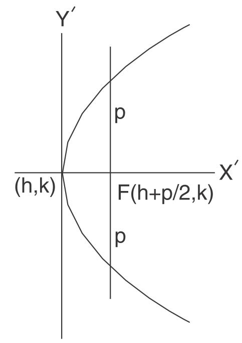
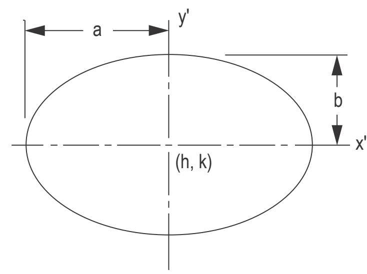


$(y - k)^2 = 2p(x - h)$; Center at (h, k)
is the standard form of the equation. When $h = k = 0$,
Focus: $(p/2, 0)$; Directrix: $x = -p/2$

Case 2. Ellipse $e < 1$:



$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1; \text{ Center at } (h, k)$$

is the standard form of the equation. When $h = k = 0$,

Eccentricity: $e = \sqrt{1 - (b^2/a^2)} = c/a$

$b = a\sqrt{1 - e^2}$;

Focus: $(\pm ae, 0)$; Directrix: $x = \pm a/e$

♦ Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publishing, 1967.
• Brink, R.W., *A First Year of College Mathematics*, D. Appleton-Century Co., Inc., 1937.

Case 3. Hyperbola $e > 1$:

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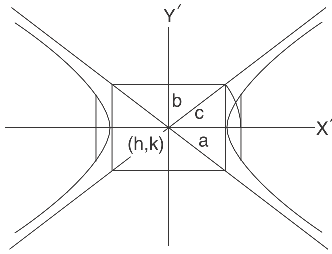

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1; \text{ Center at } (h, k)$$

is the standard form of the equation. When $h = k = 0$,

Eccentricity: $e = \sqrt{1 + \left(b^2/a^2\right)} = c/a$

$b = a\sqrt{e^2 - 1};$

Focus: $(\pm\, ae, 0)$; Directrix: $x = \pm\, a/e$

Case 4. Circle $e = 0$:

$(x - h)^2 + (y - k)^2 = r^2$; Center at (h, k) is the standard form of the equation with radius

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

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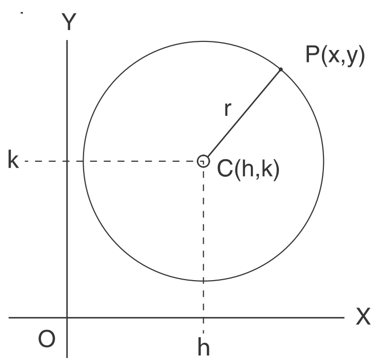

Length of the tangent line from a point on a circle to a point (x',y'):

$$t^2 = (x' - h)^2 + (y' - k)^2 - r^2$$

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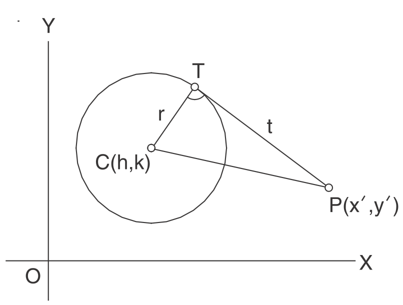

• Brink, R.W., *A First Year of College Mathematics*, D. Appleton-Century Co., Inc., 1937.

Conic Section Equation

The general form of the conic section equation is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where not both A and C are zero.

If $B^2 - 4AC < 0$, an *ellipse* is defined.

If $B^2 - 4AC > 0$, a *hyperbola* is defined.

If $B^2 - 4AC = 0$, the conic is a *parabola*.

If $A = C$ and $B = 0$, a *circle* is defined.

If $A = B = C = 0$, a *straight line* is defined.

$$x^2 + y^2 + 2ax + 2by + c = 0$$

is the normal form of the conic section equation, if that conic section has a principal axis parallel to a coordinate axis.

$h = -a$; $k = -b$

$r = \sqrt{a^2 + b^2 - c}$

If $a^2 + b^2 - c$ is positive, a *circle*, center $(-a, -b)$.

If $a^2 + b^2 - c$ equals zero, a *point* at $(-a, -b)$.

If $a^2 + b^2 - c$ is negative, locus is *imaginary*.

DIFFERENTIAL CALCULUS

The Derivative

For any function $y = f(x)$,

the derivative $= D_x y = dy/dx = y'$

$$y' = \lim_{\Delta x \to 0} \left[(\Delta y)/(\Delta x)\right]$$

$$= \lim_{\Delta x \to 0} \left\{\left[f(x + \Delta x) - f(x)\right]/(\Delta x)\right\}$$

$$y' = \text{ the slope of the curve } f(x).$$

Test for a Maximum

$y = f(x)$ is a maximum for

$x = a$, if $f'(a) = 0$ and $f''(a) < 0$.

Test for a Minimum

$y = f(x)$ is a minimum for

$x = a$, if $f'(a) = 0$ and $f''(a) > 0$.

Test for a Point of Inflection

$y = f(x)$ has a point of inflection at $x = a$,

if $f''(a) = 0$, and

if $f''(x)$ changes sign as x increases through $x = a$.

The Partial Derivative

In a function of two independent variables x and y, a derivative with respect to one of the variables may be found if the other variable is *assumed* to remain constant. If *y is kept fixed*, the function

$$z = f(x, y)$$

becomes a function of the *single variable x*, and its derivative (if it exists) can be found. This derivative is called the *partial derivative of z with respect to x*. The partial derivative with respect to x is denoted as follows:

$$\frac{\partial z}{\partial x} = \frac{\partial f(x, y)}{\partial x}$$

The Curvature of Any Curve

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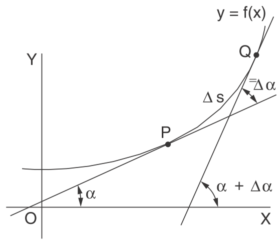


The curvature K of a curve at P is the limit of its average curvature for the arc PQ as Q approaches P. This is also expressed as: the curvature of a curve at a given point is the rate-of-change of its inclination with respect to its arc length.

$$K = \lim_{\Delta s \to 0} \frac{\Delta\alpha}{\Delta s} = \frac{d\alpha}{ds}$$

Curvature in Rectangular Coordinates

$$K = \frac{y''}{\left[1 + (y')^2\right]^{3/2}}$$

When it may be easier to differentiate the function with respect to y rather than x, the notation x' will be used for the derivative.

$$x' = dx/dy$$

$$K = \frac{-x''}{\left[1 + (x')^2\right]^{3/2}}$$

The Radius of Curvature

The *radius of curvature R* at any point on a curve is defined as the absolute value of the reciprocal of the curvature K at that point.

$$R = \frac{1}{|K|} \qquad (K \neq 0)$$

$$R = \left|\frac{\left[1 + (y')^2\right]^{3/2}}{|y''|}\right| \qquad (y'' \neq 0)$$

L'Hospital's Rule (L'Hôpital's Rule)

If the fractional function $f(x)/g(x)$ assumes one of the indeterminate forms 0/0 or ∞/∞ (where α is finite or infinite), then

$$\lim_{x \to \alpha} f(x)/g(x)$$

is equal to the first of the expressions

$$\lim_{x \to \alpha} \frac{f'(x)}{g'(x)}, \lim_{x \to \alpha} \frac{f''(x)}{g''(x)}, \lim_{x \to \alpha} \frac{f'''(x)}{g'''(x)}$$

which is not indeterminate, provided such first indicated limit exists.

INTEGRAL CALCULUS

The definite integral is defined as:

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)\,dx$$

Also, $\Delta x_i \to 0$ for all i.

A table of derivatives and integrals is available in the Derivatives and Indefinite Integrals section. The integral equations can be used along with the following methods of integration:

A. Integration by Parts (integral equation #6),
B. Integration by Substitution, and
C. Separation of Rational Fractions into Partial Fractions.

♦ Wade, Thomas L., *Calculus*, Ginn & Company/Simon & Schuster Publishers, 1953.

DERIVATIVES AND INDEFINITE INTEGRALS

In these formulas, u, v, and w represent functions of x. Also, a, c, and n represent constants. All arguments of the trigonometric functions are in radians. A constant of integration should be added to the integrals. To avoid terminology difficulty, the following definitions are followed: arcsin $u = \sin^{-1} u$, $(\sin u)^{-1} = 1/\sin u$.

1. $dc/dx = 0$
2. $dx/dx = 1$
3. $d(cu)/dx = c\, du/dx$
4. $d(u + v - w)/dx = du/dx + dv/dx - dw/dx$
5. $d(uv)/dx = u\, dv/dx + v\, du/dx$
6. $d(uvw)/dx = uv\, dw/dx + uw\, dv/dx + vw\, du/dx$
7. $\dfrac{d(u/v)}{dx} = \dfrac{v\, du/dx - u\, dv/dx}{v^2}$
8. $d(u^n)/dx = nu^{n-1}\, du/dx$
9. $d[f(u)]/dx = \{d[f(u)]/du\}\, du/dx$
10. $du/dx = 1/(dx/du)$
11. $\dfrac{d(\log_a u)}{dx} = (\log_a e)\dfrac{1}{u}\dfrac{du}{dx}$
12. $\dfrac{d(\ln u)}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$
13. $\dfrac{d(a^u)}{dx} = (\ln a)a^u \dfrac{du}{dx}$
14. $d(e^u)/dx = e^u\, du/dx$
15. $d(u^v)/dx = vu^{v-1}\, du/dx + (\ln u)\, u^v\, dv/dx$
16. $d(\sin u)/dx = \cos u\, du/dx$
17. $d(\cos u)/dx = -\sin u\, du/dx$
18. $d(\tan u)/dx = \sec^2 u\, du/dx$
19. $d(\cot u)/dx = -\csc^2 u\, du/dx$
20. $d(\sec u)/dx = \sec u \tan u\, du/dx$
21. $d(\csc u)/dx = -\csc u \cot u\, du/dx$
22. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{1}{\sqrt{1 - u^2}}\dfrac{du}{dx} \qquad (-\pi/2 \le \sin^{-1} u \le \pi/2)$
23. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{1}{\sqrt{1 - u^2}}\dfrac{du}{dx} \qquad (0 \le \cos^{-1} u \le \pi)$
24. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{1}{1 + u^2}\dfrac{du}{dx} \qquad (-\pi/2 < \tan^{-1} u < \pi/2)$
25. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{1}{1 + u^2}\dfrac{du}{dx} \qquad (0 < \cot^{-1} u < \pi)$
26. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{1}{u\sqrt{u^2 - 1}}\dfrac{du}{dx}$

    $(0 < \sec^{-1} u < \pi/2)(-\pi \le \sec^{-1} u < -\pi/2)$
27. $\dfrac{d(\csc^{-1} u)}{dx} = -\dfrac{1}{u\sqrt{u^2 - 1}}\dfrac{du}{dx}$

    $(0 < \csc^{-1} u \le \pi/2)(-\pi < \csc^{-1} u \le -\pi/2)$

1. $\int d\, f(x) = f(x)$
2. $\int dx = x$
3. $\int a\, f(x)\, dx = a \int f(x)\, dx$
4. $\int [u(x) \pm v(x)]\, dx = \int u(x)\, dx \pm \int v(x)\, dx$
5. $\int x^m dx = \dfrac{x^{m+1}}{m + 1} \qquad (m \ne -1)$
6. $\int u(x)\, dv(x) = u(x)\, v(x) - \int v(x)\, du(x)$
7. $\int \dfrac{dx}{ax + b} = \dfrac{1}{a}\ln|ax + b|$
8. $\int \dfrac{dx}{\sqrt{x}} = 2\sqrt{x}$
9. $\int a^x dx = \dfrac{a^x}{\ln a}$
10. $\int \sin x\, dx = -\cos x$
11. $\int \cos x\, dx = \sin x$
12. $\int \sin^2 x dx = \dfrac{x}{2} - \dfrac{\sin 2x}{4}$
13. $\int \cos^2 x dx = \dfrac{x}{2} + \dfrac{\sin 2x}{4}$
14. $\int x \sin x\, dx = \sin x - x \cos x$
15. $\int x \cos x\, dx = \cos x + x \sin x$
16. $\int \sin x \cos x\, dx = (\sin^2 x)/2$
17. $\int \sin ax \cos bx\, dx = -\dfrac{\cos(a - b)x}{2(a - b)} - \dfrac{\cos(a + b)x}{2(a + b)} \; (a^2 \ne b^2)$
18. $\int \tan x\, dx = -\ln|\cos x| = \ln|\sec x|$
19. $\int \cot x\, dx = -\ln|\csc x| = \ln|\sin x|$
20. $\int \tan^2 x\, dx = \tan x - x$
21. $\int \cot^2 x\, dx = -\cot x - x$
22. $\int e^{ax}\, dx = (1/a)\, e^{ax}$
23. $\int xe^{ax}\, dx = (e^{ax}/a^2)(ax - 1)$
24. $\int \ln x\, dx = x[\ln(x) - 1] \qquad (x > 0)$
25. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} \qquad (a \ne 0)$
26. $\int \dfrac{dx}{ax^2 + c} = \dfrac{1}{\sqrt{ac}}\tan^{-1}\left(x\sqrt{\dfrac{a}{c}}\right), \qquad (a > 0, c > 0)$

27a. $\int \dfrac{dx}{ax^2 + bx + c} = \dfrac{2}{\sqrt{4ac - b^2}}\tan^{-1}\dfrac{2ax + b}{\sqrt{4ac - b^2}}$

$(4ac - b^2 > 0)$

27b. $\int \dfrac{dx}{ax^2 + bx + c} = \dfrac{1}{\sqrt{b^2 - 4ac}}\ln\left|\dfrac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}}\right|$

$(b^2 - 4ac > 0)$

27c. $\int \dfrac{dx}{ax^2 + bx + c} = -\dfrac{2}{2ax + b}, \qquad (b^2 - 4ac = 0)$

CENTROIDS AND MOMENTS OF INERTIA

The *location of the centroid of an area*, bounded by the axes and the function $y = f(x)$, can be found by integration.

$$x_c = \frac{\int xdA}{A}$$

$$y_c = \frac{\int ydA}{A}$$

$$A = \int f(x)dx$$

$$dA = f(x)dx = g(y)dy$$

The *first moment of area* with respect to the y-axis and the x-axis, respectively, are:

$$M_y = \int x\, dA = x_c A$$

$$M_x = \int y\, dA = y_c A$$

The *moment of inertia* (*second moment of area*) with respect to the y-axis and the x-axis, respectively, are:

$$I_y = \int x^2\, dA$$

$$I_x = \int y^2\, dA$$

The moment of inertia taken with respect to an axis passing through the area's centroid is the *centroidal moment of inertia*. The *parallel axis theorem* for the moment of inertia with respect to another axis parallel with and located d units from the centroidal axis is expressed by

$$I_{\text{parallel axis}} = I_c + Ad^2$$

In a plane, $J = \int r^2 dA = I_x + I_y$

PROGRESSIONS AND SERIES

Arithmetic Progression

To determine whether a given finite sequence of numbers is an arithmetic progression, subtract each number from the following number. If the differences are equal, the series is arithmetic.

1. The first term is a.
2. The common difference is d.
3. The number of terms is n.
4. The last or nth term is l.
5. The sum of n terms is S.

$$l = a + (n - 1)d$$

$$S = n(a + l)/2 = n\,[2a + (n - 1)\,d]/2$$

Geometric Progression

To determine whether a given finite sequence is a geometric progression (G.P.), divide each number after the first by the preceding number. If the quotients are equal, the series is geometric:

1. The first term is a.
2. The common ratio is r.
3. The number of terms is n.
4. The last or nth term is l.
5. The sum of n terms is S.

$$l = ar^{n-1}$$

$$S = a\,(1 - r^n)/(1 - r);\ r \neq 1$$

$$S = (a - rl)/(1 - r);\ r \neq 1$$

$$\lim_{n\to\infty} S_n = a/(1 - r);\ r < 1$$

A G.P. converges if $|r| < 1$ and it diverges if $|r| > 1$.

Properties of Series

$$\sum_{i=1}^{n} c = nc; \qquad c = \text{constant}$$

$$\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} (x_i + y_i - z_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} z_i$$

$$\sum_{x=1}^{n} x = (n + n^2)/2$$

Power Series

$$\sum_{i=0}^{\infty} a_i (x - a)^i$$

1. A power series, which is convergent in the interval $-R < x < R$, defines a function of x that is continuous for all values of x within the interval and is said to represent the function in that interval.
2. A power series may be differentiated term by term within its interval of convergence. The resulting series has the same interval of convergence as the original series (except possibly at the end points of the series).
3. A power series may be integrated term by term provided the limits of integration are within the interval of convergence of the series.
4. Two power series may be added, subtracted, or multiplied, and the resulting series in each case is convergent, at least, in the interval common to the two series.
5. Using the process of long division (as for polynomials), two power series may be divided one by the other within their common interval of convergence.

Taylor's Series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \ldots$$

is called *Taylor's series*, and the function $f(x)$ is said to be expanded about the point a in a Taylor's series.

If $a = 0$, the Taylor's series equation becomes a *Maclaurin's series*.

DIFFERENTIAL EQUATIONS

A common class of ordinary linear differential equations is

$$b_n \frac{d^n y(x)}{dx^n} + \ldots + b_1 \frac{dy(x)}{dx} + b_0 y(x) = f(x)$$

where $b_n, \ldots, b_i, \ldots, b_1, b_0$ are constants.

When the equation is a homogeneous differential equation, $f(x) = 0$, the solution is

$$y_h(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \ldots + C_i e^{r_i x} + \ldots + C_n e^{r_n x}$$

where r_n is the nth distinct root of the characteristic polynomial $P(x)$ with

$$P(r) = b_n r^n + b_{n-1} r^{n-1} + \ldots + b_1 r + b_0$$

If the root $r_1 = r_2$, then $C_2e^{r_2x}$ is replaced with $C_2xe^{r_1x}$.

Higher orders of multiplicity imply higher powers of x. The complete solution for the differential equation is

$$y(x) = y_h(x) + y_p(x),$$

where $y_p(x)$ is any particular solution with $f(x)$ present. If $f(x)$ has e^{r_nx} terms, then resonance is manifested. Furthermore, specific $f(x)$ forms result in specific $y_p(x)$ forms, some of which are:

$f(x)$	$y_p(x)$
A	B
$Ae^{\alpha x}$	$Be^{\alpha x}, \alpha \neq r_n$
$A_1 \sin \omega x + A_2 \cos \omega x$	$B_1 \sin \omega x + B_2 \cos \omega x$

If the independent variable is time t, then transient dynamic solutions are implied.

First-Order Linear Homogeneous Differential Equations with Constant Coefficients

$y' + ay = 0$, where a is a real constant:

Solution, $y = Ce^{-at}$

where C = a constant that satisfies the initial conditions.

First-Order Linear Nonhomogeneous Differential Equations

$$\tau \frac{dy}{dt} + y = Kx(t) \qquad x(t) = \begin{Bmatrix} A & t < 0 \\ B & t > 0 \end{Bmatrix}$$

$$y(0) = KA$$

τ is the time constant

K is the gain

The solution is

$$y(t) = KA + (KB - KA)\left(1 - \exp\left(\frac{-t}{\tau}\right)\right) \text{ or}$$

$$\frac{t}{\tau} = 1\text{n}\left[\frac{KB - KA}{KB - y}\right]$$

Second-Order Linear Homogeneous Differential Equations with Constant Coefficients

An equation of the form

$$y'' + ay' + by = 0$$

can be solved by the method of undetermined coefficients where a solution of the form $y = Ce^{rx}$ is sought. Substitution of this solution gives

$$(r^2 + ar + b)\, Ce^{rx} = 0$$

and since Ce^{rx} cannot be zero, the characteristic equation must vanish or

$$r^2 + ar + b = 0$$

The roots of the characteristic equation are

$$r_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

and can be real and distinct for $a^2 > 4b$, real and equal for $a^2 = 4b$, and complex for $a^2 < 4b$.

If $a^2 > 4b$, the solution is of the form (overdamped)

$$y = C_1e^{r_1x} + C_2e^{r_2x}$$

If $a^2 = 4b$, the solution is of the form (critically damped)

$$y = (C_1 + C_2x)e^{r_1x}$$

If $a^2 < 4b$, the solution is of the form (underdamped)

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x), \text{ where}$$

$$\alpha = -a/2$$

$$\beta = \frac{\sqrt{4b - a^2}}{2}$$

FOURIER TRANSFORM

The Fourier transform pair, one form of which is

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$$

$$f(t) = [1/(2\pi)] \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega$$

can be used to characterize a broad class of signal models in terms of their frequency or spectral content. Some useful transform pairs are:

$f(t)$	$F(\omega)$
$\delta(t)$	1
$u(t)$	$\pi\delta(\omega) + 1/j\omega$
$u\left(t + \frac{\tau}{2}\right) - u\left(t - \frac{\tau}{2}\right) = r_{rect}\frac{t}{\tau}$	$\tau\frac{\sin(\omega\tau/2)}{\omega\tau/2}$
$e^{j\omega_o t}$	$2\pi\delta(\omega - \omega_o)$

Some mathematical liberties are required to obtain the second and fourth form. Other Fourier transforms are derivable from the Laplace transform by replacing s with $j\omega$ provided

$$f(t) = 0, t < 0$$

$$\int_0^{\infty} |f(t)| dt < \infty$$

FOURIER SERIES

Every periodic function $f(t)$ which has the period $T = 2\pi/\omega_0$ and has certain continuity conditions can be represented by a series plus a constant

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

The above holds if $f(t)$ has a continuous derivative $f'(t)$ for all t. It should be noted that the various sinusoids present in the series are orthogonal on the interval 0 to T and as a result the coefficients are given by

$$a_0 = (1/T)\int_0^T f(t)dt$$

$$a_n = (2/T)\int_0^T f(t)\cos(n\omega_0 t)dt \qquad n = 1,2,\ldots$$

$$b_n = (2/T)\int_0^T f(t)\sin(n\omega_0 t)dt \qquad n = 1,2,\ldots$$

The constants a_n and b_n are the *Fourier coefficients* of $f(t)$ for the interval 0 to T and the corresponding series is called the *Fourier series of* $f(t)$ over the same interval.

The integrals have the same value when evaluated over any interval of length T.

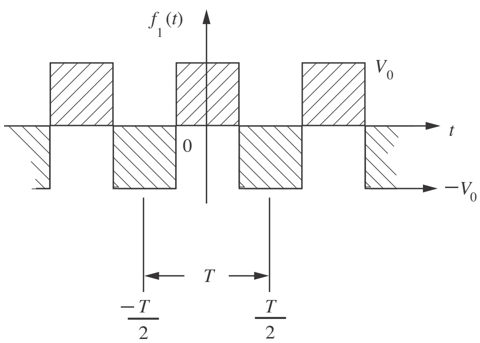
If a Fourier series representing a periodic function is truncated after term $n = N$ the mean square value F_N^2 of the truncated series is given by Parseval's relation. This relation says that the mean-square value is the sum of the mean-square values of the Fourier components, or

$$F_N^2 = a_0^2 + (1/2)\sum_{n=1}^{N}\left(a_n^2 + b_n^2\right)$$

and the RMS value is then defined to be the square root of this quantity or F_N.

Three useful and common Fourier series forms are defined in terms of the following graphs (with $\omega_0 = 2\pi/T$).

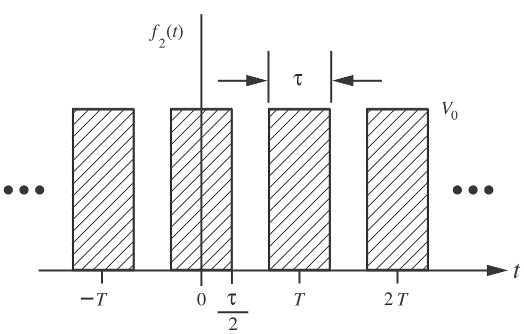
Given:



then

$$f_1(t) = \sum_{\substack{n=1 \\ (\text{n odd})}}^{\infty}(-1)^{(n-1)/2}\left(4V_0/n\pi\right)\cos\left(n\omega_0 t\right)$$

Given:

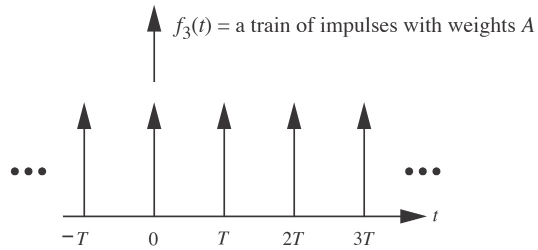


then

$$f_2(t) = \frac{V_0\tau}{T} + \frac{2V_0\tau}{T}\sum_{n=1}^{\infty}\frac{\sin(n\pi\tau/T)}{(n\pi\tau/T)}\cos\left(n\omega_0 t\right)$$

$$f_2(t) = \frac{V_0\tau}{T}\sum_{n=-\infty}^{\infty}\frac{\sin(n\pi\tau/T)}{(n\pi\tau/T)}e^{jn\omega_0 t}$$

Given:



then

$$f_3(t) = \sum_{n=-\infty}^{\infty} A\delta(t - nT)$$

$$f_3(t) = (A/T) + (2A/T)\sum_{n=1}^{\infty}\cos\left(n\omega_0 t\right)$$

$$f_3(t) = (A/T)\sum_{n=-\infty}^{\infty}e^{jn\omega_0 t}$$

The Fourier Transform and its Inverse

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi ft}\,dt$$

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi ft}\,df$$

We say that $x(t)$ and $X(f)$ form a *Fourier transform pair*:

$$x(t) \leftrightarrow X(f)$$

Fourier Transform Pairs

$x(t)$	$X(f)$
1	$\delta(f)$
$\delta(t)$	1
$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\Pi(t/\tau)$	$\tau\,\text{sinc}(\tau f)$
$\text{sinc}(Bt)$	$\frac{1}{B}\Pi(f/B)$
$\Lambda(t/\tau)$	$\tau\,\text{sinc}^2(\tau f)$
$e^{-at}u(t)$	$\frac{1}{a+j2\pi f} \quad a>0$
$te^{-at}u(t)$	$\frac{1}{(a+j2\pi f)^2} \quad a>0$
$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2+(2\pi f)^2} \quad a>0$
$e^{-(at)^2}$	$\frac{\sqrt{\pi}}{a}e^{-(\pi f/a)^2}$
$\cos(2\pi f_0t+\theta)$	$\frac{1}{2}[e^{j\theta}\delta(f-f_0)+e^{-j\theta}\delta(f+f_0)]$
$\sin(2\pi f_0t+\theta)$	$\frac{1}{2j}[e^{j\theta}\delta(f-f_0)-e^{-j\theta}\delta(f+f_0)]$
$\sum_{n=-\infty}^{n=+\infty}\delta(t-nT_s)$	$f_s\sum_{k=-\infty}^{k=+\infty}\delta(f-kf_s) \quad f_s=\frac{1}{T_s}$

Fourier Transform Theorems

Linearity	$ax(t)+by(t)$	$aX(f)+bY(f)$
Scale change	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{f}{a}\right)$
Time reversal	$x(-t)$	$X(-f)$
Duality	$X(t)$	$x(-f)$
Time shift	$x(t-t_0)$	$X(f)e^{-j2\pi ft_0}$
Frequency shift	$x(t)e^{j2\pi f_0t}$	$X(f-f_0)$
Modulation	$x(t)\cos 2\pi f_0t$	$\frac{1}{2}X(f-f_0)$ $+\frac{1}{2}X(f+f_0)$
Multiplication	$x(t)y(t)$	$X(f)*Y(f)$
Convolution	$x(t)*y(t)$	$X(f)Y(f)$
Differentiation	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
Integration	$\int_{-\infty}^{t}x(\lambda)d\lambda$	$\frac{1}{j2\pi f}X(f)$ $+\frac{1}{2}X(0)\delta(f)$

where:

$$\text{sinc}(t)=\frac{\sin(\pi t)}{\pi t}$$

$$\Pi(t)=\begin{cases}1, & \lvert t\rvert\le\frac{1}{2}\\ 0, & \text{otherwise}\end{cases}$$

$$\Lambda(t)=\begin{cases}1-\lvert t\rvert, & \lvert t\rvert\le 1\\ 0, & \text{otherwise}\end{cases}$$

LAPLACE TRANSFORMS

The unilateral Laplace transform pair

$$F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} dt$$

where $s = \sigma + j\omega$

represents a powerful tool for the transient and frequency response of linear time invariant systems. Some useful Laplace transform pairs are:

$f(t)$	$F(s)$
$\delta(t)$, Impulse at $t = 0$	1
$u(t)$, Step at $t = 0$	$1/s$
$t[u(t)]$, Ramp at $t = 0$	$1/s^2$
$e^{-\alpha t}$	$1/(s + \alpha)$
$te^{-\alpha t}$	$1/(s + \alpha)^2$
$e^{-\alpha t} \sin \beta t$	$\beta/[(s + \alpha)^2 + \beta^2]$
$e^{-\alpha t} \cos \beta t$	$(s + \alpha)/[(s + \alpha)^2 + \beta^2]$
$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - \sum_{m=0}^{n-1} s^{n-m-1} \frac{d^m f(0)}{dt^m}$
$\int_0^t f(\tau) d\tau$	$(1/s)F(s)$
$\int_0^\infty x(t - \tau) h(\tau) d\tau$	$H(s)X(s)$
$f(t - \tau)\ u(t - \tau)$	$e^{-\tau s} F(s)$
$\lim_{t \to \infty} f(t)$	$\lim_{s \to 0} sF(s)$
$\lim_{t \to 0} f(t)$	$\lim_{s \to \infty} sF(s)$

The last two transforms represent the Final Value Theorem (F.V.T.) and Initial Value Theorem (I.V.T.), respectively. It is assumed that the limits exist.

MATRICES

A matrix is an ordered rectangular array of numbers with m rows and n columns. The element a_{ij} refers to row i and column j.

Multiplication of Two Matrices

$$A = \begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix} \quad A_{3,2} \text{ is a 3-row, 2-column matrix}$$

$$B = \begin{bmatrix} H & I \\ J & K \end{bmatrix} \quad B_{2,2} \text{ is a 2-row, 2-column matrix}$$

In order for multiplication to be possible, the number of columns in A must equal the number of rows in B.

Multiplying matrix B by matrix A occurs as follows:

$$C = \begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix} \cdot \begin{bmatrix} H & I \\ J & K \end{bmatrix}$$

$$C = \begin{bmatrix} (A \cdot H + B \cdot J) & (A \cdot I + B \cdot K) \\ (C \cdot H + D \cdot J) & (C \cdot I + D \cdot K) \\ (E \cdot H + F \cdot J) & (E \cdot I + F \cdot K) \end{bmatrix}$$

Matrix multiplication is not commutative.

Addition

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} + \begin{bmatrix} G & H & I \\ J & K & L \end{bmatrix} = \begin{bmatrix} A + G & B + H & C + I \\ D + J & E + K & F + L \end{bmatrix}$$

Identity Matrix

The matrix $\mathbf{I} = (a_{ij})$ is a square $n \times n$ matrix with 1's on the diagonal and 0's everywhere else.

Matrix Transpose

Rows become columns. Columns become rows.

$$A = \begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \quad A^T = \begin{bmatrix} A & D \\ B & E \\ C & F \end{bmatrix}$$

Inverse $[\ ]^{-1}$

The inverse $\boldsymbol{B}$ of a square $n \times n$ matrix $\boldsymbol{A}$ is

$$\boldsymbol{B} = \boldsymbol{A}^{-1} = \frac{\text{adj}(\boldsymbol{A})}{|\boldsymbol{A}|}, \text{ where}$$

adj($\boldsymbol{A}$) = adjoint of $\boldsymbol{A}$ (obtained by replacing $\boldsymbol{A}^T$ elements with their cofactors) and $|\boldsymbol{A}|$= determinant of $\boldsymbol{A}$.

$[A][A]^{-1} = [A]^{-1}[A] = [I]$ where **I** is the identity matrix.

DETERMINANTS

A *determinant of order n* consists of n^2 numbers, called the *elements* of the determinant, arranged in *n* rows and *n* columns and enclosed by two vertical lines.

In any determinant, the *minor* of a given element is the determinant that remains after all of the elements are struck out that lie in the same row and in the same column as the given element. Consider an element which lies in the *j*th column and the *i*th row. The *cofactor* of this element is the value of the minor of the element (if $i + j$ is *even*), and it is the negative of the value of the minor of the element (if $i + j$ is *odd*).

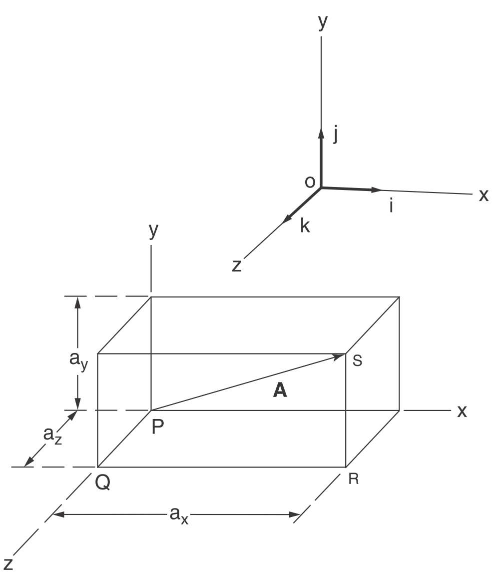
If *n* is greater than 1, the *value* of a determinant of order *n* is the sum of the *n* products formed by multiplying each element of some specified row (or column) by its cofactor. This sum is called the *expansion of the determinant* [according to the elements of the specified row (or column)]. For a second-order determinant:

$$\begin{vmatrix} a_1 a_2 \\ b_1 b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

For a third-order determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$$

VECTORS



$$\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

Addition and *subtraction:*

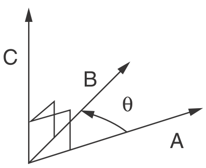
$$\mathbf{A} + \mathbf{B} = (a_x + b_x)\mathbf{i} + (a_y + b_y)\mathbf{j} + (a_z + b_z)\mathbf{k}$$
$$\mathbf{A} - \mathbf{B} = (a_x - b_x)\mathbf{i} + (a_y - b_y)\mathbf{j} + (a_z - b_z)\mathbf{k}$$

The *dot product* is a *scalar product* and represents the projection of **B** onto **A** times $|\mathbf{A}|$. It is given by

$$\mathbf{A}\bullet\mathbf{B} = a_x b_x + a_y b_y + a_z b_z$$
$$= |\mathbf{A}||\mathbf{B}|\cos\theta = \mathbf{B}\bullet\mathbf{A}$$

The *cross product* is a *vector product* of magnitude $|\mathbf{B}|\,|\mathbf{A}|\sin\theta$ which is perpendicular to the plane containing **A** and **B**. The product is

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = -\,\mathbf{B}\times\mathbf{A}$$



The sense of $\mathbf{A}\times\mathbf{B}$ is determined by the right-hand rule.

$$\mathbf{A}\times\mathbf{B} = |\mathbf{A}||\mathbf{B}|\mathbf{n}\sin\theta, \text{ where}$$

$\mathbf{n}$ = unit vector perpendicular to the plane of **A** and **B**.

Gradient, Divergence, and Curl

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\phi$$

$$\nabla\cdot\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}\right)$$

$$\nabla\times\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\times\left(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}\right)$$

The Laplacian of a scalar function ϕ is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Identities

$\mathbf{A}\bullet\mathbf{B} = \mathbf{B}\bullet\mathbf{A}$; $\mathbf{A}\bullet(\mathbf{B}+\mathbf{C}) = \mathbf{A}\bullet\mathbf{B} + \mathbf{A}\bullet\mathbf{C}$

$\mathbf{A}\bullet\mathbf{A} = |\mathbf{A}|^2$

$\mathbf{i}\bullet\mathbf{i} = \mathbf{j}\bullet\mathbf{j} = \mathbf{k}\bullet\mathbf{k} = 1$

$\mathbf{i}\bullet\mathbf{j} = \mathbf{j}\bullet\mathbf{k} = \mathbf{k}\bullet\mathbf{i} = 0$

If $\mathbf{A}\bullet\mathbf{B} = 0$, then either $\mathbf{A} = 0$, $\mathbf{B} = 0$, or **A** is perpendicular to **B**.

$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$

$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = (\mathbf{A}\times\mathbf{B}) + (\mathbf{A}\times\mathbf{C})$

$(\mathbf{B}+\mathbf{C})\times\mathbf{A} = (\mathbf{B}\times\mathbf{A}) + (\mathbf{C}\times\mathbf{A})$

$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$

$\mathbf{i}\times\mathbf{j} = \mathbf{k} = -\mathbf{j}\times\mathbf{i}$; $\mathbf{j}\times\mathbf{k} = \mathbf{i} = -\mathbf{k}\times\mathbf{j}$

$\mathbf{k}\times\mathbf{i} = \mathbf{j} = -\mathbf{i}\times\mathbf{k}$

If $\mathbf{A}\times\mathbf{B} = \mathbf{0}$, then either $\mathbf{A} = \mathbf{0}$, $\mathbf{B} = \mathbf{0}$, or **A** is parallel to **B**.

$$\nabla^2\phi = \nabla\cdot(\nabla\phi) = (\nabla\cdot\nabla)\phi$$

$$\nabla\times\nabla\phi = \mathbf{0}$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

DIFFERENCE EQUATIONS

Any system whose input $v(t)$ and output $y(t)$ are defined only at the equally spaced intervals

$$f(t) = y' = \frac{y_{i+1} - y_i}{t_{i+1} - t_i}$$

can be described by a difference equation.

First-Order Linear Difference Equation

$$\Delta t = t_{i+1} - t_i$$

$$y_{i+1} = y_i + y'(\Delta t)$$

NUMERICAL METHODS

Newton's Method for Root Extraction

Given a function $f(x)$ which has a simple root of $f(x) = 0$ at $x = a$ an important computational task would be to find that root. If $f(x)$ has a continuous first derivative then the $(j+1)$st estimate of the root is

$$a^{j+1} = a^j - \left.\frac{f(x)}{\frac{df(x)}{dx}}\right|_{x = a^j}$$

The initial estimate of the root a^0 must be near enough to the actual root to cause the algorithm to converge to the root.

Newton's Method of Minimization

Given a scalar value function

$$h(\boldsymbol{x}) = h(x_1, x_2, \dots, x_n)$$

find a vector $\boldsymbol{x}^* \in R_n$ such that

$$h(\boldsymbol{x}^*) \le h(\boldsymbol{x}) \text{ for all } \boldsymbol{x}$$

Newton's algorithm is

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \left(\left.\frac{\partial^2 h}{\partial x^2}\right|_{\boldsymbol{x} = \boldsymbol{x}_k}\right)^{-1} \left.\frac{\partial h}{\partial x}\right|_{\boldsymbol{x} = \boldsymbol{x}_k}, \text{ where}$$

$$\frac{\partial h}{\partial x} = \begin{bmatrix} \frac{\partial h}{\partial x_1} \\ \frac{\partial h}{\partial x_2} \\ \dots \\ \dots \\ \frac{\partial h}{\partial x_n} \end{bmatrix}$$

and

$$\frac{\partial^2 h}{\partial x^2} = \begin{bmatrix} \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 h}{\partial x_1 \partial x_2} & \dots & \dots & \frac{\partial^2 h}{\partial x_1 \partial x_n} \\ \frac{\partial^2 h}{\partial x_1 \partial x_2} & \frac{\partial^2 h}{\partial x_2^2} & \dots & \dots & \frac{\partial^2 h}{\partial x_2 \partial x_n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial^2 h}{\partial x_1 \partial x_n} & \frac{\partial^2 h}{\partial x_2 \partial x_n} & \dots & \dots & \frac{\partial^2 h}{\partial x_n^2} \end{bmatrix}$$

Numerical Integration

Three of the more common numerical integration algorithms used to evaluate the integral

$$\int_a^b f(x)\,dx$$

are:

Euler's or Forward Rectangular Rule

$$\int_a^b f(x)\,dx \approx \Delta x \sum_{k=0}^{n-1} f(a + k\Delta x)$$

Trapezoidal Rule

for $n = 1$

$$\int_a^b f(x)\,dx \approx \Delta x\left[\frac{f(a) + f(b)}{2}\right]$$

for $n > 1$

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(a) + 2\sum_{k=1}^{n-1} f(a + k\Delta x) + f(b)\right]$$

Simpson's Rule/Parabolic Rule (n must be an even integer)

for $n = 2$

$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

for $n \ge 4$

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}\begin{bmatrix} f(a) + 2\sum\limits_{k=2,4,6,\dots}^{n-2} f(a + k\Delta x) \\ +4\sum\limits_{k=1,3,5,\dots}^{n-1} f(a + k\Delta x) + f(b) \end{bmatrix}$$

with $\Delta x = (b - a)/n$

n = number of intervals between data points

Numerical Solution of Ordinary Differential Equations

<u>Euler's Approximation</u>

Given a differential equation

$$dx/dt = f(x, t) \text{ with } x(0) = x_o$$

At some general time $k\Delta t$

$$x[(k+1)\Delta t] \cong x(k\Delta t) + \Delta t f[x(k\Delta t), k\Delta t]$$

which can be used with starting condition x_o to solve recursively for $x(\Delta t), x(2\Delta t), \dots, x(n\Delta t)$.

The method can be extended to nth order differential equations by recasting them as n first-order equations.

In particular, when $dx/dt = f(x)$

$$x[(k+1)\Delta t] \cong x(k\Delta t) + \Delta t f[x(k\Delta t)]$$

which can be expressed as the recursive equation

$$x_{k+1} = x_k + \Delta t\,(dx_k/dt)$$

$$x_{k+1} = x + \Delta t\,[f(x(k), t(k))]$$