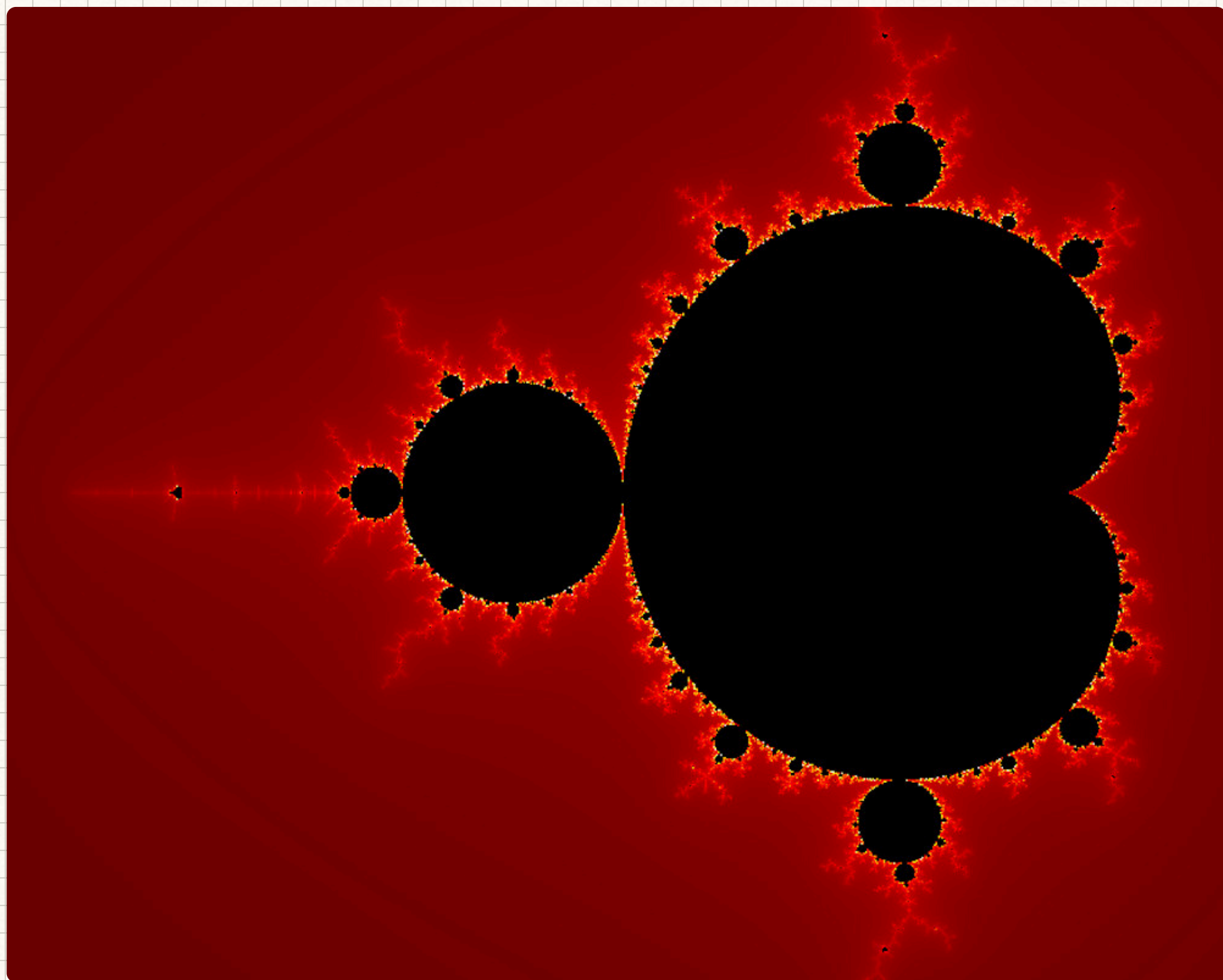




Course Introduction

1. Preliminaries -

- Roster + Survey (only 3 Q's?)
- Intro to Me w/ Chaos (Moro + Kennedy)
- What is Chaos Theory
- Web Page and Syllabus and PDF Notes
- Structure of the Course - Relative History and No Text book

2. What is Chaos Theory? -

- Survey - Chaos, Fractals, SOC
- Jurassic Park and Other Introductions
- Where fits in science and math - Philosophical

3. Powerpoint Introduction -



DEPARTMENT OF MATHEMATICS AND STATISTICS
College of Science and Engineering

WINONA
STATE UNIVERSITY

The Beauty of Chaos

乱腾



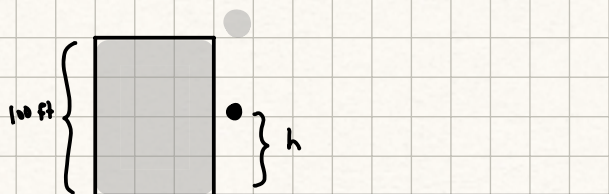
Chaos in Context

1. Basic Definitions -

- Deterministic - state of the system can be completely determined from:

- ① Initial State
- ② A Set of Law governing the System

Ex: Falling Body



Initial State:

$$h(0) = 100 \text{ ft}$$

$$v(0) = -8 \frac{\text{ft}}{\text{s}}$$

Laws Governing System: (Newton's)

$$a = -32 \frac{\text{ft}}{\text{s}^2} \quad (\text{Gravity, 2}^{\text{nd}} \text{ Law of Motion})$$

State of System:

$$a = -32$$

$$v' = -32$$

$$v = -32t + C$$

$$v(0) = -32(0) + C = C = -8$$

$$v = -32t - 8$$

$$h' = -32t - 8$$

$$h = -16t^2 - 8t + D$$

$$h(0) = -16(0)^2 - 8(0) + D = D = 100$$

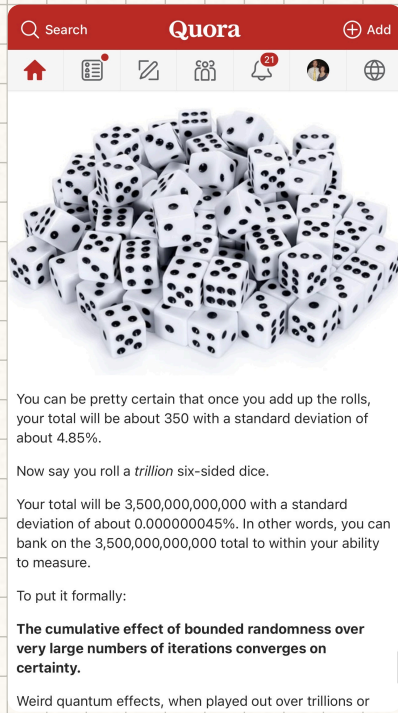
$$\therefore h(t) = -16t^2 - 8t + 100.$$

- Random - the state of the system cannot be determined from its initial state and a set of physical laws; i.e. the same initial state can give rise to different behaviors.

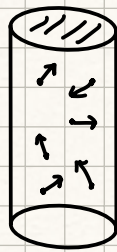
Non-Ex: Rolling a 6-sided die

- Use language of probability

- But this is deterministic ; can write physical laws governing behavior
- Probability allows us to uncover a global pattern w/o even calculating the fate of a specific die.



Non-Ex: Kinetic Theory of Gases



$$n = 6.02 \times 10^{23}$$

$$P \propto T$$

$$P = kT \longrightarrow k = \frac{P}{T}$$

proportionality constant

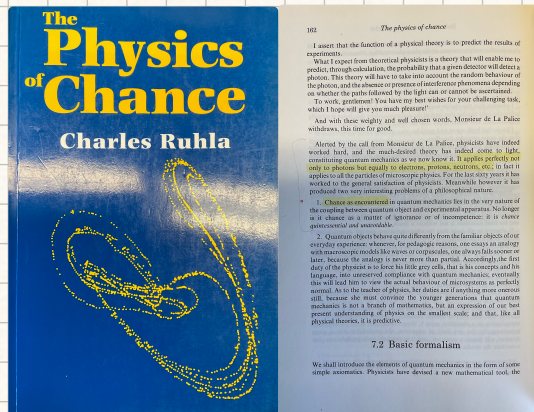
Closed Container

- volume of container
- amount of gas
- type of gas
- other stuff

Ex: Have to use the language of probability, because there is no underlying mechanisms causing the behavior.

So, probability is not being used to by-pass complex calculations.

Quantum Mechanics





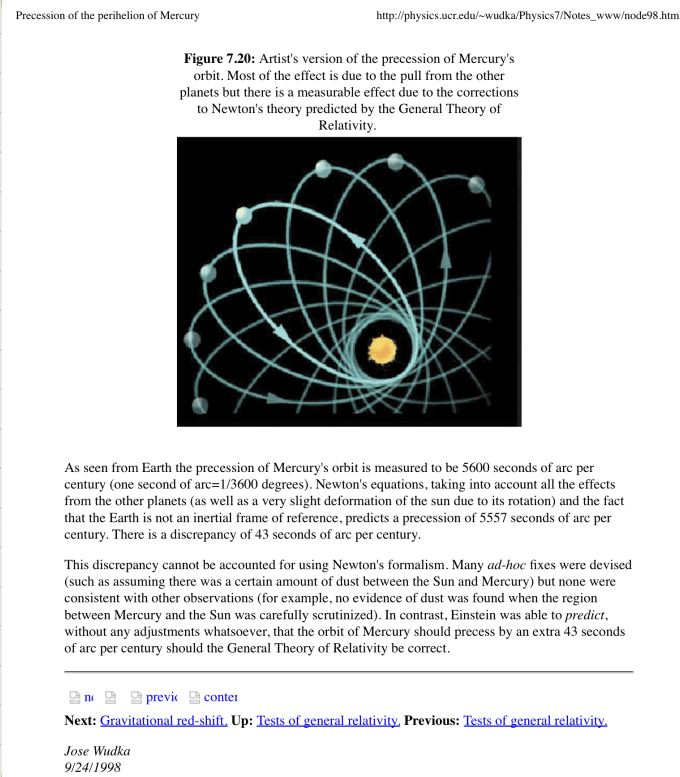
Regarding how Quantum Theory can only be true if it is, at its foundation, probabilistic:

"The problem is this: At first glance, wave-particle duality is not only mysterious but inconsistent in a blatant way. The inconsistency can be understood with a thought experiment. Imagine a burst of light from which a light wave ripples out through an ever-widening sphere in space. As the wave travels, it gets more attenuated, since the energy in it is getting spread over a wider and wider area. (That is why the farther you are from a light bulb, the fainter it appears.) Now, suppose a light-collecting device is set up, a box with a shutter-essentially, a camera. The farther away it is placed from the light burst, the less light it will collect. Suppose the light-collecting box is set up at a distance where it will collect exactly a thousandth of the light emitted in the burst. The inconsistency arises if the original burst contained, say, fifty particles of light. For then it appears that the light-collector must have collected 0.05 particles (a thousandth of fifty), which is impossible, since particles of light are indivisible. A wave, being continuous, can be infinitely attenuated or subdivided, whereas a particle cannot.

"Quantum theory resolves this by saying that the light-collector, rather than collecting 0.05 particles, has a 0.05 probability of collecting one particle. More precisely, the average number of particles it will collect, if the same experiment is repeated many times, is 0.05. Wave-particle duality, which gave rise to quantum theory in the first place, forces us to accept that quantum physics is inherently probabilistic. Roughly speaking, in pre-quantum, classical physics, one calculated what actually happens, while in quantum physics one calculates the relative probabilities of various things happening." (Barr, p.3-4)

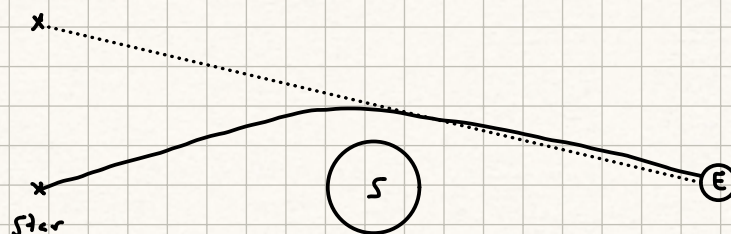
Things get even stranger, and it is clear why some people pine for the good old days when waves were waves and particles were particles. One of those people was Albert Einstein. He detested the idea that a fundamental theory should yield only probabilities. "God does not play dice!" he insisted. In Einstein's view, the need for probabilities simply showed that the theory was incomplete. History supported his claim, for in classical physics the use of probabilities always stemmed from incomplete information. For example, if one says that there is a 60 percent chance of a baseball hitting a glass window, it is only because one doesn't know the ball's direction and speed well enough. If one knew them better (and also knew the wind velocity and all other relevant variables), one could definitely say whether the ball would hit the window. For Einstein, the probabilities in quantum theory meant only that there were as-yet-unknown variables: hidden variables, as they are called. If these were known, then in principle everything could be predicted exactly, as in classical physics. (Barr, p.4)

- Statistical Model - reveals patterns in a set of data, but gives no insight into the underlying mechanisms causing the behavior.
- Ex: Young Male Drivers have higher insurance rates. (< 25 yrs.)
- Ex: Precession of Mercury's Orbit

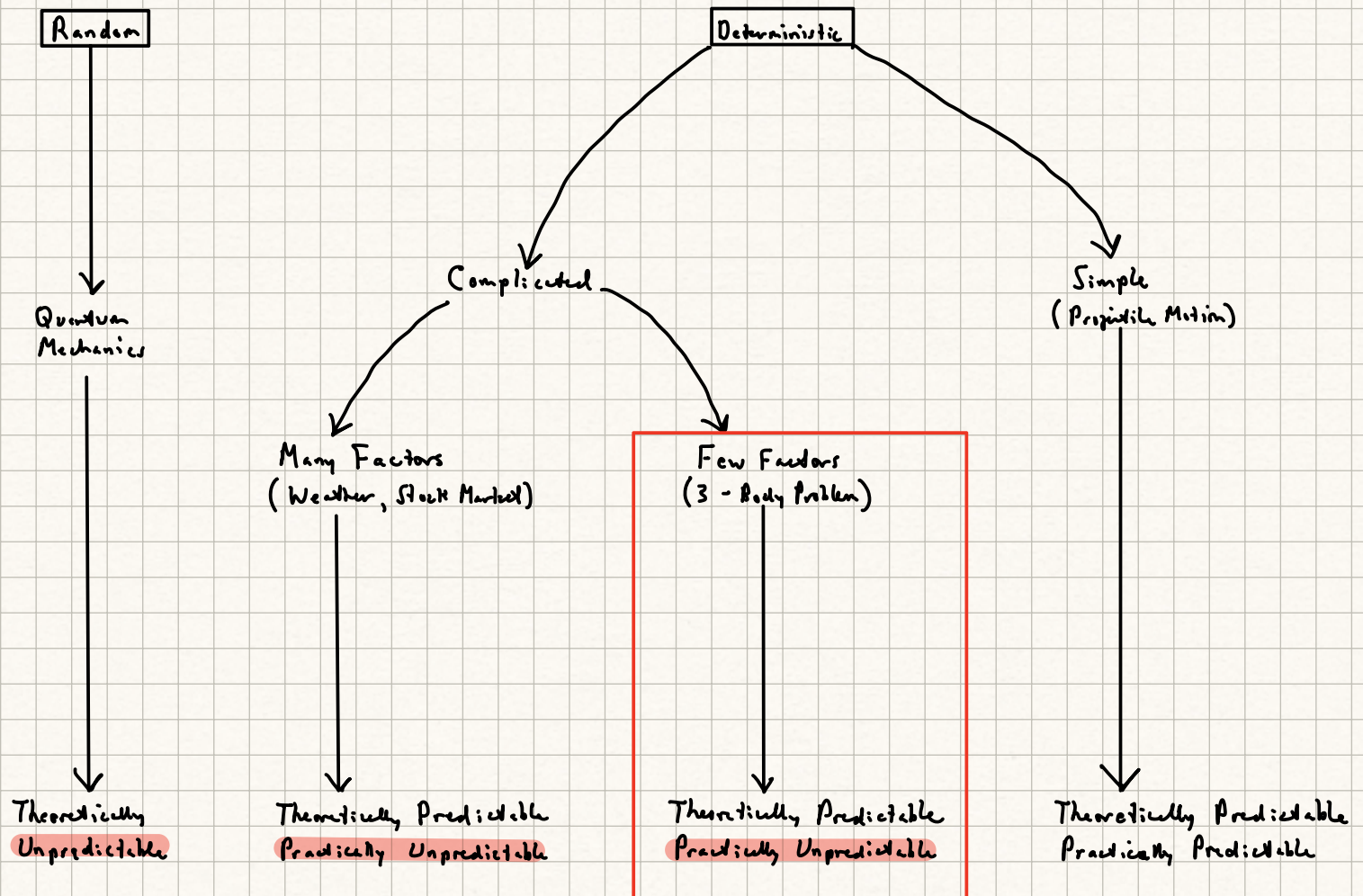


- Theoretical Model - are formulated from first principle (foundational observation of the universe; physical laws) and describe the evolution of a system from a given state. Give insight into the underlying mechanisms causing the behavior.

- Ex: Young Male Drivers - Why?: testosterone, social conditioning, frontal lobe development
- Ex: Einstein's Theory of Relativity - gravity is the effect of mass warping space.



2. Conceptual Framework -



<https://www.wired.com/story/technology-philosophy-three-city-problem/>

NASA CAN LAND a probe on Saturn's largest moon, 764 million miles from Earth—yet no one has been able to mathematically demonstrate the exact positions of the Earth, sun, and our own moon at a given point in the future. Scientists can make estimates, but these all rely on simplifications.

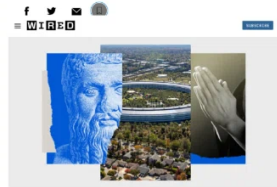


PHOTO ILLUSTRATION: NIKO STAFF, GETTY IMAGES

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Two-body problems, like mapping the movement of one planet around one star, are solvable. These binary orbits are easy to predict. But a serious complication arises if a third body is introduced. Our moon, which has the gravitational forces of both the sun and the Earth acting upon it simultaneously, is part of an infamous three-body problem.

Trying to solve for the movement of three large bodies in one another's orbit creates a circular logic. The calculations rely on the initial positions of the three bodies, but these initial positions are unknowable over time because the bodies always affect one another in unpredictable ways. In the 300 years since Isaac Newton outlined the dilemma in his Principia, diligent physicists have only been able to offer special-case solutions for restricted versions of the problem. "In a nonlinear system like the chaotic three-body problem," writes Caroline Delbert for Popular Mechanics, "all bets are off, and our intuitions are scrambled."

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On the Difference Between Using Probabilities to Negotiate Incomplete Knowledge and Envisioning Probability at the Foundation of a Theory:

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On Classical Physics as Deterministic:

Classical physics was strictly deterministic, so that (as Laplace famously said) if the state of the physical world were completely specified at one instant, its whole future development would be exactly and uniquely determined. Whether a man lifts his arm or nods his head now would (in a world governed by classical physical laws) be an inevitable consequence of the state of the world a billion years ago. (Barr, p. 5)

Free Will?

On the Revolution of Chaos Theory:

“As one physicist put it: ‘Relativity eliminated the Newtonian illusion of absolute space and time; quantum theory eliminated the Newtonian dream of a controllable measurement process; and chaos eliminates the Laplacian fantasy of deterministic predictability.’ Of the three, the revolution in chaos applies to the universe we see and touch, to objects at human scale. Everyday experience and real pictures of the world have become legitimate targets for inquiry. There has long been a feeling, not always expressed openly, that theoretical physics has strayed far from human intuition about the world. Whether this will prove to be fruitful heresy or just plain heresy, no one knows. But some of those who thought physics might be working its way into a corner now look to chaos as a way out.” (Gleick, p.3)

On The Essence of Chaos Theory:

“You might argue that the reason for this unpredictability is that there are simply too many variables present in meteorological or economic systems. That is indeed true in these cases, but this is by no means the complete answer. One of the remarkable discoveries of twentieth-century mathematics is that very simple systems, even systems depending on only one variable, may behave just as unpredictably as the stock market, just as wildly as a turbulent waterfall, and just as violently as a hurricane. The culprit, the reason for this unpredictable behavior, has been called “chaos” by mathematicians.” (Devaney, p. 2)

On the Relationship of Chaos Theory to the World on a Human Scale:

Gleick summed up the revolutionary effect of Yorke’s paper: “Yorke had offered more than mathematical result. He had sent a message to physicists: Chaos is ubiquitous; it is stable; it is structured.” (Yorke, p. 2)

Introduction to Modeling

1. Model - a mathematical expression of a phenomenon, which is built on underlying principles.

Ex:



Hooke's Law says that the force exerted by a spring is proportional to its distortion.

$$F \propto x$$

$$F = kx$$

Proportionality Constant
(Parameter)

What affects the value of k ?

[List all factors affecting F]

- x {
 - Distortion
- k {
 - Material
 - Temperature
 - Coiling
 - Thickness of Coiling Material
 - Other stuff?

x cannot affect k if the model is to remain valid.

The model ceases to be valid if any of the factors that affect k change.

How do we find the value of k ? Empirically.

$$F = kx$$

$$k = \frac{F}{x}$$

x	F	$k = \frac{F}{x}$
2	6	3
0.5	1.5	3
3	9	3

} constant

$$\therefore k = 3$$

Ex:

Force of air resistance on an object is proportional to the velocity of the object relative to the air.

$$F \propto v$$

$$F = kv$$

What affects this value?

[List all factors affecting F]

- v {
- v
 - cross-sectional area
 - spin
 - density of air
- k {
- texture
 - altitude
 - humidity
 - orientation

v	F	$k = \frac{F}{v}$
2	8	4
3	18	6
1	2	2
4	32	8

} not constant

$\therefore$ model does not fit the data.

k is not constant; it depends on v .

In fact, $k = 2v$

$$\therefore F = kv = (2v)v$$

$$F = 2v^2$$

$$F \propto v^2$$

Ex: Newton's Laws of Motion: $F = ma$ (2nd)

Newton's Law of Gravity: the force of gravity between 2 objects is jointly proportional to their mass, m and M , and inversely proportional to the square of the distance between their centers of mass.

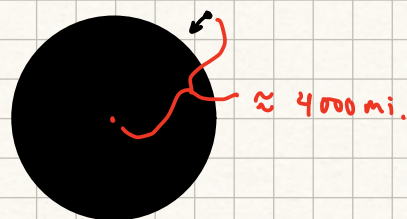
$$F \propto m \cdot M \cdot \frac{1}{d^2}$$

$$F = G \cdot m \cdot M \cdot \frac{1}{d^2}$$

$$F = \frac{G m M}{d^2}$$

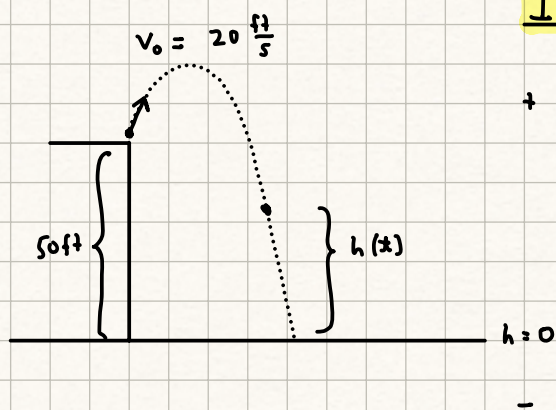
$$F = ma$$

$$a = \frac{GM}{d^2} = 32 \frac{ft}{s^2} \approx 9.81 \frac{m}{s^2}$$



Introduction to Dynamical Systems

1. Ex:



$$a = -32 \frac{\text{ft}}{\text{s}^2}$$

$$v' = -32$$

$$\left. \begin{array}{l} v = -32t + C \\ v(0) = 20 \end{array} \right\} -32(0) + C = 20 \rightarrow C = 20$$

$$v = -32t + 20$$

$$h' = -32t + 20$$

$$\left. \begin{array}{l} h = -32 \cdot \frac{1}{2} t^2 + 20t + D \\ h(0) = 50 \end{array} \right\} -16(0)^2 + 20(0) + D = 50 \rightarrow D = 50$$

$$h = -16t^2 + 20t + 50$$

$$\begin{array}{l} \text{Differential Egn.} \\ \text{Initial Conditions} \end{array} \left\{ \begin{array}{l} h'' = -32 \\ h'(0) = 20 \\ h(0) = 50 \end{array} \right\} \text{Continuous Dynamical System (CDS)}$$

2. Dynamical System - any system (model, equation) that depends on time, either implicitly or explicitly.

- dynamic - depends on time.
- static - time independent (e.g. $PV = nRT$)
- explicit - dependence on time is directly indicated in the equation (e.g. $h = -16t^2 + 20t + 50$)
- implicit - exact nature of the dependence on time is hidden - implied ($h'' = -32$)
- continuous - iff time can be any real number. [CDS]
- discrete - iff time can take on only integer values. [DDS]

3. Ex:

Malthusian Model of Population Growth

ROC of a population $P \propto P$

$$\frac{dP}{dt} = kP$$

What affects this value?

- Population
- Cost of Living
- Disease
- Fertility
- ✓ Food
- Wars
- Predators
- Climate
- M/F Ratio
- Species
- Life Expectancy
- Geography
- ✓ Space
- Natural Disasters
- Immigration
- Emigration
- Other

Suppose $k = 3$.

implicit

$$\frac{dP}{dt} = 3P$$

vs. linear

initial population

$$P = Ae^{3t}$$

growth/death rate

$$\frac{dP}{dt} = 3t$$

$$P = \frac{3}{2} t^2 + C$$

$$\frac{dP}{dt} = Ae^{3t} \cdot 3 = P \cdot 3$$

$$P = Ae^{3t} \leftarrow \text{explicit (solution)}$$

4. Ex: Verhulst Model

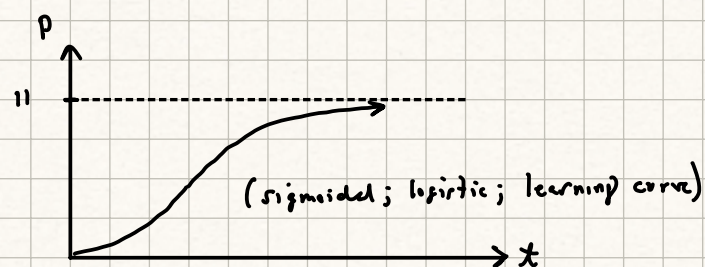
$$\frac{dP}{dt} = k \left(1 - \frac{P}{C} \right) P$$

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{C} \right)$$

Suppose $K = 3$ and $C = 11$

$$\frac{dP}{dt} = 3P \left(1 - \frac{P}{11} \right) \leftarrow \text{implicit (non-linear)}$$

$$P = \frac{11e^{3t}}{e^{3t} + D} \leftarrow \text{explicit}$$



$$\frac{dP}{dt} = 3P \left(1 - \frac{P}{c}\right) + t \quad \leftarrow \text{no solution (algebraically)}$$

5. Discrete Dynamical Systems - (DDS)

Ex: Malthusian Model

CDS	DDS
Implicit $\left\{ \begin{array}{l} \frac{dP}{dt} = 0.78P \\ P(0) = 50 \end{array} \right.$ Linear	Implicit Form (Difference Eq.) $\left\{ \begin{array}{l} \frac{\Delta P}{\Delta t} = 0.78P \\ P_0 = 50 \end{array} \right.$ Linear
	$\frac{P_n - P_{n-1}}{n - (n-1)} = 0.78 P_{n-1}$
	$P_n - P_{n-1} = 0.78 P_{n-1}$
	$P_n = P_{n-1} + 0.78 P_{n-1}$
	Implicit (Recursion) $\left\{ \begin{array}{l} P_n = 1.78 P_{n-1} \\ P_0 = 50 \end{array} \right.$
	$P_1 = 1.78 \cdot 50$
	$P_2 = 1.78 \cdot P_1 = 1.78 (1.78 \cdot 50) = 1.78^2 \cdot 50$
	$P_3 = 1.78 \cdot P_2 = 1.78 (1.78^2 \cdot 50) = 1.78^3 \cdot 50$
	$\vdots$
	$P_n = 1.78^n \cdot 50 \quad \text{Explicit Form (Sequence)}$
$P(t) = 50 e^{0.78t} = 50 \cdot 2.18^t$	

Note: Any DDS with an explicit form (solution) is well-behaved (clearly defined and easily predictable at any time n). If a DDS has a closed form, it cannot be chaotic.

Discrete Dynamic Systems

1. Notation - we identify a DDS by its recursion rule.

Ex:

$$\left. \begin{array}{l} P_n = 1.78 P_{n-1} \\ P_0 = 50 \end{array} \right\}$$

$$f(x) = 1.78x \quad \leftarrow \text{recursion rule}$$

$$f(0) = 50 \quad \leftarrow \text{initial condition}$$

$$x_0 = 50$$

$$x_1 = f(x_0) = 1.78 \cdot 50$$

$$x_2 = f(x_1) = f(f(x_0)) = f^2(x_0)$$

$\vdots$

$$x_n = f^n(x_0) \quad \left[= \overbrace{f(f(f(\dots f(x_0))))}^{n\text{-times}} \right]$$

Observation: In general, if $f(x) = mx$, then $x_n = m^n \cdot x_0$ (exponential)
 $\uparrow$
Linear

Ex:

$$\left. \begin{array}{l} f(x) = 3x + 1 \\ x_0 = 50 \end{array} \right\} \text{ DDS}$$

$\nwarrow$ affine

$$x_0 = 50$$

$$x_1 = 3(50) + 1$$

$$x_2 = 3x_1 + 1 = 3(3 \cdot 50 + 1) + 1$$

$$x_3 = 3x_2 + 1 = 3[3(3 \cdot 50 + 1) + 1] + 1$$

$\vdots$

$$x_n = ? \quad (\text{PS 2})$$

Ex:

$$f(x) = x^3 - 3x$$

$$x_0 = 2 \quad ; \quad \dots \quad x_n = 2$$

$$x_0 = -2 \quad ; \quad \dots \quad x_n = -2$$

$$x_0 = 0 \quad ; \quad \dots \quad x_n = 0$$

$$x_0 = 1 \quad ; \quad x_1 = -2, x_2 = -2, \dots, x_n = -2$$

$$x_0 = -1 ; \quad x_1 = 2 ; \quad x_2 = 2, \dots, x_n = 2$$

$$x_0 = 2.01 ; \quad x_1 = 2.0906, \quad x_2 = 2.1654, \dots, \rightarrow \infty$$

$$x_0 = 1.99 ; \quad x_1 = 1.9106, \quad x_2 = 1.84763, \rightarrow ?$$