# Homework on Mathematical Modeling

## Mathematics 302—Chaos Theory

## Dr. Peratt

1. For each of the following scenarios, write the *form* of equation which would model the relationship between the given quantities. Also, determine some factors that could affect the value of the proportionality constant and some which cannot affect its value if the model is to remain valid.

   (a) The strength, $S$ of a beam is proportional to the square of its thickness, $h$.

   (b) The frequency of a pendulum $F$ is inversely proportional to the square root of the length of the arm $L$.

   (c) The current $I$ in a wire is proportional to the voltage $V$ and inversely proportional to the resistance $R$ in the wire.

   (d) The energy, $E$, expended by a swimming dolphin is proportional to the cube of its speed, $v$.

   (e) The velocity of a wave in a string ($v$) is proportional to the square root of the tension ($T$) and inversely proportional to the density ($\rho$).

   (f) The intensity of a sound wave ($I$) is inversely proportional to the square of its distance $d$ from the source.

   (g) The frequency of vibration of a string ($F$) is proportional to the square root of the tension in the string ($T$) and inversely proportional to the length ($L$) and the square root of the density ($\rho$).

   (h) When two bodies undergo a direct collision, the difference in their velocities ($v_1$ and $v_2$) immediately after the impact is proportional to the difference in their velocities ($u_1$ and $u_2$) immediately before impact. The proportionality constant is typically labeled $e$. (Taken from *Basic Biomechanics*, 6th Edition by Susan Hall, p. 400. This observation was originally made by Newton, and the proportionality constant is called the *coefficient of restitution*).

   (i) For every 3 feet an object lies below the ocean, the pressure exerted on the object increases by 1.2 pounds per square inch. When the object sits on top of the water, the pressure on it is 14.5 pounds per square inch.

   (j) The force, $F$, exerted by a spring is proportional to the amount of deviation, $x$, from its natural length.

   (k) At high velocities, the force of air resistance, $F$, is proportional to the square of the velocity, $v$, of the moving object.

2. Exercise in Allometry:

   (a) For a given organism (or object), reason that $M \propto V$, where $M$ represents the mass of the object and $V$ its volume. What, exactly, does the proportionality constant represent in this case.

   (b) Now, we reason that $S \propto V^p$, for some power of $p$, where $S$ represents the surface area of the object and $V$ its volume. Use analysis of measurement dimensions to make a reasonable conjecture about the value of $p$. Also, give an intuitive interpretation for the proportionality constant in this case.

(c) Now, combine the above two results to find a relationship between $S$ and $M$.

(d) We now note that, a human body of mass 70 kg has a surface area, on average, of approximately 18,000 square centimeters. Find the constant of proportionality for humans. Then, use the formula to estimate the surface area of a human with a body mass of 87 kg.

3. The number of species of lizard, $N$, found on an island off Baja California is proportional to the fourth root of the area, $A$, of the island. Write a formula for $N$ as a function of $A$. Find the value of the proportionality constant if the population in an area of 2 square miles is estimated to be 130. Also, determine some factors that could affect the value of the proportionality constant and some which cannot affect its value.

4. The specific heat, $s$, of an element is the number of calories of heat required to raise the temperature of one gram of the element by one degree Centigrade. Use the following table to decide if $s$ is proportional or inversely proportional to the atomic weight, $w$, of the element. Find the proportionality constant.

| Element | Li | Mg | Al | Fe | Ag | Pb | Hg |
|---|---|---|---|---|---|---|---|
| $w$ | 6.9 | 24.3 | 27.0 | 55.8 | 107.9 | 207.2 | 200.6 |
| $s$ | 0.92 | 0.25 | 0.21 | 0.11 | 0.056 | 0.031 | 0.033 |

5. For each variable, $p_i$, in the data table below, determine whether it is proportional to $t$, $t^2$, $t^3$, inversely proportional to $t$, or none of these.

| $t$ | 10.0 | 20.0 | 30.0 | 40.0 | 50.0 |
|---|---|---|---|---|---|
| $p_1$ | 2.7 | 5.4 | 8.1 | 10.8 | 13.5 |
| $p_2$ | 540.0 | 4320.0 | 14580.0 | 34560.0 | 67500.0 |
| $p_3$ | 26.0 | 49.0 | 72.0 | 95.0 | 118.0 |
| $p_4$ | 25.0 | 60.0 | 100.0 | 140.0 | 200.0 |
| $p_5$ | 0.5 | 0.3 | 0.2 | 0.1 | 0.1 |

6. For each of the problems below: (1) describe how one would compute the proportionality constant, (2) identify various factors that might affect the proportionality constant and those which cannot affect the proportionality constant, and (3) identify some conditions under which the model may not be valid.

(a) The time rate of change of a population $P$ is proportional to the square root of $P$.

(b) The time rate of change of the velocity $v$ of a coasting motorboat is proportional to the square of $v$.

(c) The acceleration $\frac{dv}{dt}$ of an automobile is propotional to the difference between 250 $\frac{km}{hr}$ and the velocity of the car. Also, what does 250 represent physically?

(d) In a city having a fixed population $P$, the time rate of change of the number $N$ of those persons who have heard a certain rumor is proportional to the number of those who have not yet heard the rumor. Also, according to the model, under what conditions will the rumor not spread? Why does this make sense?

(e) In a city with a fixed population $P$, the time rate of change of the number $N$ of those persons infected with a certain contagious disease is jointly proportional to the number of those who have the disease *and* the number who do not have the disease. Also, according to the model, under what conditions will the disease not spread? Why does this make sense?

# Homework on Mathematical Modeling—Solutions

Mathematics 302—Chaos Theory

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1. For each of the following scenarios, write the *form* of equation which would model the relationship between the given quantities. Also, determine some factors that could affect the value of the proportionality constant and some which cannot affect its value if the model is to remain valid.

   (a) The strength, $S$ of a bean is proportional to the square of its thickness, $h$.

   **Answer:** $S \propto h^2$, so that $S = kh^2$. The thickness cannot affect the value of $k$, but the type of wood, the age and condition of the wood, the moisture content of the wood, perhaps even the width of the wood, might affect the value of $k$.

   (b) The frequency of a pendulum $F$ is inversely proportional to the square root of the length of the arm.

   **Answer:** $F = k \cdot \dfrac{1}{\sqrt{L}}$. The length of the arm cannot affect the value of $k$, but the force of gravity, friction in the pivoting joint, air resistance, and mass of the weight at the end of the arm could affect $k$.

   (c) The current $I$ in a wire is proportional to the voltage $V$ and inversely proportional to the resistance $R$ in the wire.

   **Answer:** $I = k \cdot \dfrac{V}{R}$. The voltage and resistance cannot affect the value of $k$. It is difficult to think of factors that would affect $k$, except things like temperature and the material out of which the wire is composed – but these would affect the resistance directly. In fact, it turns out that $k = 1$, meaning that voltage and resistance are the only factors affecting the current.

   (d) The energy, $E$, expended by a swimming dolphin is proportional to the cube of its speed, $v$.

   **Answer:** $E = kv^3$. The velocity of the dolphin cannot affect the value of $k$, but the health, age, and sup-species of dolphin, the time of day, the motivation (i.e. is it being chased by a predator and is therefore expending energy in fear as well as in physical exertion), salinity and other properties of the water that may affect resistance may all affect the value of $k$.

   (e) The velocity of a wave in a string ($v$) is proportional to the square root of the tension ($T$) and inversely proportional to the density ($\rho$).

   **Answer:** $v = k \cdot \dfrac{\sqrt{T}}{\rho}$. The tension and density cannot affect the value of $k$, material out of which the string is composed (accounting for properties other than density), the type of surrounding medium (e.g. water or air), and temperature could affect it.

   (f) The intensity of a sound wave ($I$) is inversely proportional to the square of its distance $d$ from the source.

   **Answer:** $I = k \cdot \dfrac{1}{d^2}$. The distance from the source cannot affect the value of $k$, but the medium in which the sound is traveling, temperature, and the intensity with which it leave the source (i.e. "loudness") could all affect $k$.

(g) The frequency of vibration of a string ($F$) is proportional to the square root of the tension in the string ($T$) and inversely proportional to the length ($L$) and the square root of the density ($\rho$).

**Answer:** $F = k \cdot \dfrac{\sqrt{T}}{L \cdot \sqrt{\rho}}$. The tension, length, and density cannot affect $k$, but material out of which the string is composed (accounting for properties other than density), the type of surrounding medium (e.g. water or air), and temperature could affect it.

(h) When two bodies undergo a direct collision, the difference in their velocities ($v_1$ and $v_2$) immediately after the impact is proportional to the difference in their velocities ($u_1$ and $u_2$) immediately before impact. The proportionality constant is typically labeled $e$. (Taken from *Basic Biomechanics*, 6th Edition by Susan Hall, p. 400. This observation was originally made by Newton, and the proportionality constant is called the *coefficient of restitution*).

**Answer:** We have that the difference after the collision, $v_1 - v_2$ is proportional to the difference before the collision, $u_1 - u_2$. Therefore, the model is $v_1 - v_2 = e(u_1 - u_2)$. The factors that can affect the value of $e$ are anything that can affect the elasticity of the collision except for the velocities: mass of objects, material out of which the objects are made, temperature of the objects, pressure within the objects (if they are inflatable balls, for example).

(i) For every 3 feet an object lies below the ocean, the pressure exerted on the object increases by 1.2 pounds per square inch. When the object sits on top of the water, the pressure on it is 14.5 pounds per square inch.

**Answer:** $p = \frac{1.2}{3}d + 14.5$, where $d$ represents the depth of the object below the surface of the water. The constants in this equation would be, in its general form, $p = kd + n$, where the salinity and temperature of the water, for example, may affect the value of $k$, and the altitude of the body of water and the temperature of the atmosphere may affect the value of $n$, but the depth $d$ can affect neither value.

(j) The force, $F$, exerted by a spring is proportional to the amount of deviation, $x$, from its natural length.

**Answer:** $F = kx$, also referred to as Hooke's Law. While $x$ cannot affect the value of $k$ if the model is to remain valid, the material out of which the spring is composed, the tightness of the coiling, and the temperature of the material could all affect the value of $k$.

(k) At high velocities, the force of air resistance, $F$, is proportional to the square of the velocity, $v$, of the moving object.

**Answer:** $F = kv^2$. The velocity cannot affect the value of $k$, but the density of the air, the particulate matter in the air, the shape of the falling object, the temperature of the air, the level of moisture content in the air could all affect the value of $k$.

2. Exercise in Allometry:

(a) For a given organism (or object), reason that $M \propto V$, where $M$ represents the mass of the object and $V$ its volume. What, exactly, does the proportionality constant represent in this case.

**Answer:** As the volume of an object of a given substance increases, so will its mass. The formula will be $M = \rho V$, and $\rho = \frac{M}{V}$ represents the density of the object.

(b) Now, we reason that $S \propto V^p$, for some power of $p$, where $S$ represents the surface area of the object and $V$ its volume. Use analysis of measurement dimensions to make a reasonable conjecture about the value of $p$. Also, give an intuitive interpretation for the proportionality constant in this case.

**Answer:** Since volume is measured in $cm^3$ and surface area in $cm^2$, a reasonable conjecture will be that $S \propto V^{2/3}$, or $S = \beta V^{2/3}$. Since $\beta$ will primarily be affected by the shape of the object (i.e. an object that is very thin and flat will have a larger surface area than a sphere of the same volume), we can think of $\beta$ as representing primarily the shape of the object.

(c) Now, combine the above two results to find a relationship between $S$ and $M$.

**Answer:** We have $S = \alpha V^{2/3} = \alpha \left(\frac{1}{\rho} M\right)^{2/3} = \left(\alpha \cdot \left(\frac{1}{\rho}\right)^{2/3}\right) \cdot M^{2/3} = kM^{2/3}$.

So, in the end, the model will be $S = kM^{2/3}$, where $k$ will be affected by both the density and shape of the object.

(d) We now note that, a human body of mass 70 kg has a surface area, on average, of approximately 18,000 square centimeters. Find the constant of proportionality for humans. Then, use the formula to estimate the surface area of a human with a body mass of 87 kg.

**Answer:** The data given allow us to calculate $k \approx 1059.759$. Hence, a human of body mass 87 kg will have a surface area of approximately 20,808 $cm^2$.

3. The number of species of lizard, $N$, found on an island off Baja California is proportional to the fourth root of the area, $A$, of the island. Write a formula for $N$ as a function of $A$. Find the value of the proportionality constant if the population in an area of 2 square miles is estimated to be 130. Also, determine some factors that could affect the value of the proportionality constant and some which cannot affect its value.

**Answer:** We have that $N = kA^{1/4}$. Substituting in the given values gives $k = 109.3$, so that $N = 109.3A^{1/4}$. Some factors that may affect the value of $k$ are the particular climate of the area in question, the number and types of predators in the area in question, etc. Clear, the size of the area under consideration cannot affect the value of $k$ if this model is a valid one.

4. The specific heat, $s$, of an element is the number of calories of heat required to raise the temperature of one gram of the element by one degree Centigrade. Use the following table to decide if $s$ is proportional or inversely proportional to the atomic weight, $w$, of the element. Find the proportionality constant.

| Element | Li | Mg | Al | Fe | Ag | Pb | Hg |
|---|---|---|---|---|---|---|---|
| $w$ | 6.9 | 24.3 | 27.0 | 55.8 | 107.9 | 207.2 | 200.6 |
| $s$ | 0.92 | 0.25 | 0.21 | 0.11 | 0.056 | 0.031 | 0.033 |

**Answer:** If we take $s = kw$ and solve for $k$, we obtain $k = \frac{s}{w}$. If we take $s = k \cdot \frac{1}{w}$ and solve for $k$, we obtain $k = s \cdot w$. So, we simply need to compute values of $\frac{s}{w}$ and $s \cdot w$ to determine which of these is constant, if either. We see that the values of $s \cdot w$ all hover around 6 (i.e. 6.348, 6.075, 5.670, 6.138, etc.), while the values of $\frac{s}{w}$ differ markedly from each other and show a clear trend downward (i.e. 0.1333, 0.0103, 0.0078, 0.0020).

5. For each variable, $p_i$, in the data table below, determine whether it is proportional to $t$, $t^2$, $t^3$, inversely proportional to $t$, or none of these.

| $t$ | 10.0 | 20.0 | 30.0 | 40.0 | 50.0 |
|---|---|---|---|---|---|
| $p_1$ | 2.7 | 5.4 | 8.1 | 10.8 | 13.5 |
| $p_2$ | 540.0 | 4320.0 | 14580.0 | 34560.0 | 67500.0 |
| $p_3$ | 26.0 | 49.0 | 72.0 | 95.0 | 118.0 |
| $p_4$ | 25.0 | 60.0 | 100.0 | 140.0 | 200.0 |
| $p_5$ | 0.5 | 0.3 | 0.2 | 0.1 | 0.1 |

**Answer:** We see that $p_1 = 0.3t$, $p_2 = 0.54t^3$, and $p_5 = \frac{5.2}{t}$. Neither of the others is proportional to any of the given powers of $t$. In fact, $p_3 = 2.3t + 3$.

6. For each of the problems below: (1) describe how one would compute the proportionality constant, (2) identify various factors that might affect the proportionality constant and those which cannot affect the proportionality constant, and (3) identify some conditions under which the model may not be valid.

   (a) The time rate of change of a population $P$ is proportional to the square root of $P$.

   **Answer:** The model is $\frac{dP}{dt} = k\sqrt{P}$.

   - Factors that may affect the value of $k$ include: species, environmental resources (space and food), number and type of predator, incidence of disease. The population itself cannot affect $k$ if this model is to remain valid.
   - Compute $k$ by computing $\frac{dP/dt}{\sqrt{P}}$ for several different population measurements and take an average to represent the "true value" of $k$.
   - Any situation in which the value of $k$ is variable with respect to either $P$ or $t$ will invalidate the model. For example, if the food supply changes as $P$ grows, or the number of predators change with $P$ and $t$, etc., then the model will cease to be valid. Also, for some values of $P$, the rate of change may no longer be proportional to $\sqrt{P}$ but to some other power of $P$.

   (b) The time rate of change of the velocity $v$ of a coasting motorboat is proportional to the square of $v$.

   **Answer:** The model is $\frac{dv}{dt} = kv^2$.

   - Factors that may affect $k$ are the shape and mass of the boat, the density and viscosity of the water, the surface geometry of the boat, the wind speed, etc. The velocity itself is the only thing that should not affect the value of $k$.
   - We compute several values of $\frac{dv/dt}{v^2}$ for various velocities and average them. The resulting value of $k$ would like be valid only for the specific boat that we used for the calculations.
   - If any of the values that affect $k$ change with $t$ or $v$, the model would be invalidated. For example, if the wind speed changes significantly, or the viscosity of the water changes (for example, if we enters an area of warmer or dirtier water as we are coasting), this may invalidate the model.

(c) The acceleration $\frac{dv}{dt}$ of an automobile is propotional to the difference between 250 $\frac{km}{hr}$ and the velocity of the car. Also, what does 250 represent physically.

**Answer:** The model is $\frac{dv}{dt} = k(250 - v)$. When $v = 250$, the acceleration will be 0, so 250 is the maximum velocity of the car.

- The value of $k$ could depend on the shape and mass of the car, its surface geometry, the humidity, temperature, or density of the air, the wind speed and direction relative to the car, etc.
- Calculate several values of $\frac{dv/dt}{250-v}$ for several different velocities and take an average.
- Again, if anything affecting the value of $k$ changes with respect to $v$ or $t$, this would invalidate the model. An additional consideration is that the 250 represents a type of "terminal velocity," beyond which the car cannot accelerate. This value, too, may depend on factors such as the humidity, temperature, and density of the air or the wind speed—all of which can change with respect to time or velocity.

(d) In a city having a fixed population $P$, the time rate of change of the number $N$ of those persons who have heard a certain rumor is proportional to the number of those who have not yet heard the rumor. Also, according to the model, under what conditions will the rumor not spread? Why does this make sense?

**Answer:** The model is $\frac{dN}{dt} = k(P - N)$. The rumor will not spread if $P = N$, because then $\frac{dN}{dt} = 0$. This makes sense because when $P = N$, everyone will have heard the rumor.

- Factors affecting $k$ may include: the density of the population, the efficiency of the means of communication, the interest in the particular rumor, the demographics of the population (i.e. are there certain sub-segments of the population who do not communicate with others in the population?), etc.
- Compute several values of $\frac{dN/dt}{N}$ at several stages of the spreading of the rumor (practically, this may be nearly impossible to do without affecting the value of $k$ by doing it!), and take an average. We will see later on in the course a different method for computing $k$.
- Anything that makes the factors affecting $k$ change with respect to $N$ or $t$ will invalidate the model. For example, if the news media get hold of the rumor and publish it, that will affect a discontinuous change in the rate of spreading.

(e) In a city with a fixed population $P$, the time rate of change of the number $N$ of those persons infected with a ertain contagious disease is jointly proportional to the number of those who have the disease *and* the number who do not have the disease. Also, according to the model, under what conditions will the disease not spread? Why does this make sense?

**Answer:** The model is $\frac{dN}{dt} = kN(P - N)$, if we take $N$ to represent the number of people infected by the disease. The disease will not spread if $N = 0$ or $N = P$. This makes sense because if $N = 0$, no one has the disease, so it cannot spread and if $N = P$, everyone has the disease so it will not spread any more. The answers to the other questions are therefore similar, except that factors affecting $k$ will also include: the type of disease, the density of the population, the level of contact between people, etc. Therefore, some factors that may invalidate the model would be: if the disease mutates, if a quarantine is instituted or a vaccine is found and distributed, etc. These events would tend to change the level of contact between people and the communicability of the disease and would thus change the value of $k$ over time.