Problem Set #2

Mathematics 302—Chaos Theory
Dr. Peratt

Directions: Please complete the following exercises. You may confer with one another on strategy, but the spirit of the assignment is that you each learn the material and ultimately write up your own solutions either individually *or in pairs*.

1. **Affine Difference Equations:** Technically, a linear function is one that is of the form $f(x) = mx$; i.e. it maps 0 to 0. Functions of the form $f(x) = mx + b$, where $b \neq 0$, are called affine. (For folks who have taken Differential Equations, these correspond to linear DE's that are non-homogeneous, but the forcing function is just a constant.)
   (a) Find a closed form solution for the affine DS (dynamical system) defined by $f(x) = 0.83x+12.5,\ x_0 = 7.0$, and use it to find the value of the 236th term.
   (b) Repeat the above in the general case in which $f(x) = mx + b,\ x_0 = a$.
   (c) Find the closed form solution for the affine DS defined by $f(x) = x + 7$ without referring to any of the work in the previous parts of this problem. Explain why finding the closed form for this particular affine DS is uniquely simple; i.e. what problem to we *not* encounter here that we do encounter with any affine DS defined by a function of the form $f(x) = mx + b,\ m \neq 1$?

2. **Application 1–Drug Elimination:** We now consider another situation in which an affine DS is an appropriate model. Suppose that a drug is administered to a patient once per day. The initial dose is $75mg$, and each daily dose thereafter is $50mg$. Also, suppose that each day, the liver removes 24% of the drug that is in the body.
   (a) Find an affine DDS that models this situation.
   (b) Write out, not in closed form, the expression for the amount of drug present in the blood stream on the fourth day (there is nothing special about the fourth day; I just picked it). Don't simplify your answers too much on each step, or you will miss the pattern. Explain the physical significance of the descending powers.
   (c) Use iteration to approximate the long-term level of drug in the body.
   (d) Using only the implicit (or recursive) form of the DDS, determine the exact value of the long-term level of drug in the body.
   (e) Write down the closed-form of this DS, and take the limit as $n \to \infty$ to determine the exact long-term level of drug in the blood stream.

3. **Application 2–Amortization:** Amortization refers to the process of collecting compound interest on an account which receives regular payments. For example, suppose you took out a loan for \$10,000.00 at a 7.86% annual interest rate, compounded monthly. In addition, suppose that you are going to make payments of \$325.00 per month on that loan. The *principal* refers to the amount you owe at any given point. Let $P_0 = \$10,000.00$ refer to the principal at the beginning of the first month, month 0. The principal at the beginning of month one, $P1$, would be the original \$10,000.00 *plus* the interest for the first month. Since the 7.86% interest rate is annual, this comes to $\frac{7.86}{12}\%$ per month. Hence, you now owe \$10,000.00 plus $\$10,000 \cdot \frac{0.0786}{12} = \$65.50$. However, you make a payment at the end of the month, so you really owe $P_1 = \$10,000.00 + \$65.50 - \$325.00 = \$9,740.50$. You then compute P_2 in the same way, etc.
   (a) Write the function which defines the difference equation for this scenario.
   (b) Use iteration to approximate the number of months it will take to pay off the loan.
   (c) What aspect of the situation makes the equation affine?
   (d) Find a closed form for the DE.
   (e) Use the closed form to determine the number of months it will take to pay off the loan.
   (f) Use the closed form to determine what the monthly payment should be in order to pay off the loan in 3.5 years.
   (g) Use the closed form to determine what the annual rate of interest is if, after 36 months, \$4,075.89 is still owed on the loan. (This question is inspired by a true story. My sister and her husband were in financial difficulty years ago, and a "friend" offered them a personal loan for \$10,000. After 36 months, she asked him how much they still owed, and he said \$4,075.89. She called me and asked me to figure out what interest rate he was charging her. Long story short, they are no longer friends.)

4. **Closed Forms for Some Non-Linear DEs:**

   (a) For each of the following functions:

      (a) write the recursive form of the DDS defined by the function,
      (b) write the first 3 iterates on the orbit of: (1) $x_0 = 0.8$, and (2) $x_0 = 3.2$,
      (c) use iteration on your TI-89 or an Excel spreadsheet to compute x_{25} for both $x_0 = 0.8$ and $x_0 = 3.2$.
      (d) if possible, find the closed form for x_n with initial condition $x_0 = a$; if it is not possible, explain the difficulty that you encounter.

      1) $f(x) = 3x$

      2) $f(x) = x^3$

      3) $f(x) = \sqrt[3]{x}$

      4) $f(x) = \dfrac{3}{x}$

      5) $f(x) = \dfrac{x}{x+1}$

      6) $f(x) = \dfrac{1}{x+1}$

      7) $f(x) = \cos x$ (On this one, do you notice any pattern with the iterates?)

      8) $f(x) = x^2 - 1.25$ (On this one, do you notice any pattern with the iterates?)

5. **Closed Forms for Some Non-Linear DEs:**

   (a) Find the closed form for a DS based on the function $f(x) = x^2$. According to your closed form, determine what happens as you iterate your function; i.e. compute $\lim_{n\to\infty} f^n(x)$. (Recall the $f^n(x) = \overbrace{f(f(f(\dots(x))))}^{n\text{-times}}$)
   .
   (b) Repeat the above process for $f(x) = \sqrt{x}$.
   (c) Repeat the above process for $f(x) = x^2 + 1$. Explain the difficulty you encounter.