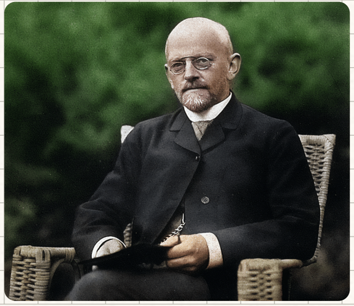
5. **Modern Work** -

- Hilbert (1862-1943)
  - Systematic Formulation of an axiomatic system for all of mathematics (Formalist School, "Principia Mathematica")
  - In contrast to Intuitionists (Constructivists, No Existence from Formal Logic Alone, × Law of Excluded Middle)
  - Euclid's Elements and added Axioms / Definitions (21 axioms vs. 5)
  - Last Mathematician to Understand All of Mathematics



- Russell (1890 - 1970)
  - Formalizing Set Theory
  - Georg Cantor showed that Set Theory is fundamental to understanding mathematics consistently.
  - Russell's Paradox

P = set of all sets

S = set of all sets that are not elements of themselves

Ex: $\{1, 5, ☺, \{1, 2, !\}, \emptyset\}$ is in S, because no element of this set is the entire set

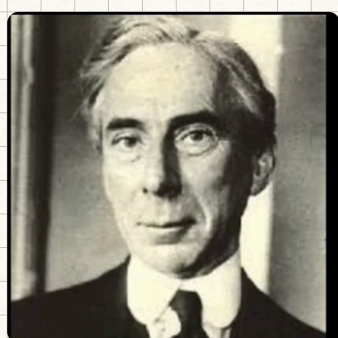
Is $S \in S$?
- If $S \in S$, then $S \notin S$.
- If $S \notin S$, then $S \in S$.

} Paradox

Concluded that our definition of set as "a collection of objects" is not precise enough, and that we must not allow certain self-referential statements.

- "There are no universal truths."
- "The only beliefs which can be established with certainty are those subject to the scientific method."
- "If God can do anything, can He make a rock so big that He can't move it?"

can do anything — omnipotent – nothing external to the being can thwart their will.



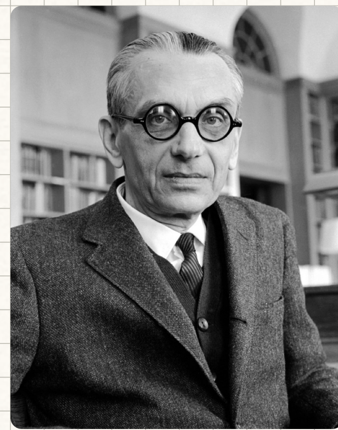
- Kurt Gödel (1906 - 1978)
  - Showed that if P is an axiomatic system complex enough to include basic arithmetic of integers, then it is possible to formulate a statement S about the consistency of P so that:

① If P is consistent, then S cannot be proven w/in the structure of P.

② If P is not consistent, then S and ~S can both be proven w/in structure of P.

∴ P is not complete since S is undecidable

- In any axiomatic system, there will be statements whose truth value cannot be established w/o adding more axioms.



Ex: Cardinality $\mathbb{Z}, \mathbb{Q}$ is $\aleph_0$

$\mathbb{R}$ is $2^{\aleph_0}$

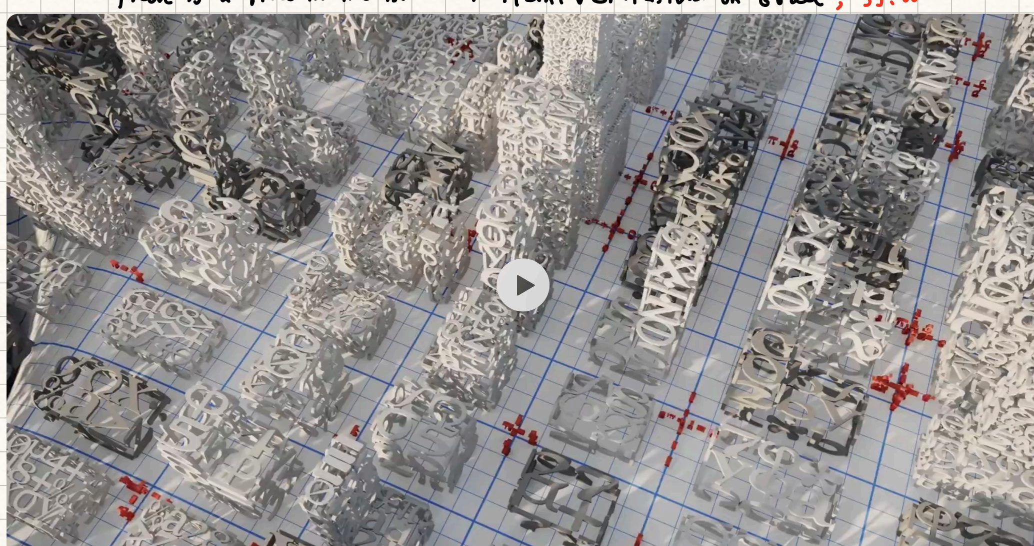
Is there a size of infinity between $\aleph_0$ and $2^{\aleph_0}$?

The answer is independent of the axioms of set theory.

Continuum Hypothesis - there is no size of infinity between $\aleph_0$ and $2^{\aleph_0}$.

Ex: Axiom of Choice - is it legitimate to choose an element from each of an uncountable number of sets to create a new set?

There is a Hole in the Bottom of Math: Veritasium on Gödel ; 33:00



Kurt Gödel

ON FORMALLY UNDECIDABLE PROPOSITIONS OF PRINCIPIA MATHEMATICA AND RELATED SYSTEMS

PREFACE

Kurt Gödel's astonishing discovery and proof, published in 1931, that even in elementary parts of arithmetic there exist propositions which cannot be proved or disproved within the system, is one of the most important contributions to logic since Aristotle. Any formal logical system which disposes of sufficient means to compass the addition and multiplication of positive integers and zero is subject to this limitation, so that one must consider this kind of incompleteness an inherent characteristic of formal mathematics as a whole, which was before this customarily considered *the* unequivocal intellectual discipline *par excellence.*

No English translation of Gödel's paper, which occupied twenty-five pages of the *Monatshefte für Mathematik und Physik*, has been generally available, and even the original German text is not everywhere easily accessible. The argument, which used a notation adapted from that of Whitehead and Russell's *Principia Mathematica*, is a closely reasoned one and the present translation—besides being a long overdue act of piety—should make it more easily intelligible and much more widely read. In the former respect the reader will be greatly aided by the Introduction contributed by the Knightbridge Professor of Moral Philosophy in the University of Cambridge; for this is an excellent work of scholarship in its own right, not only pointing out the significance of Gödel's work, but illuminating it by a paraphrase of the major part of the whole great argument.

I proposed publishing a translation after a discussion meeting on "Gödel's Theorem and its bearing on the philosophy of science", held in 1959 by the Edinburgh Philosophy 'consistent', altogether apart from whether or not t calculus is in fact interpreted to represent a deductive system.

If the calculus P were 'inconsistent', all its formulae would be 'provable' and so the condition for 'ω-consistency' (p. 21) could not be satisfied. So if P is 'ω-consistent', it is also 'consistent'. The notion of 'ω-consistency' is intimately connected with finitist methods of proof. It will not be further considered here, since it is not a necessary condition in an 'Unprovability' Theorem for Gödel's formal system P. In 1936 Rosser, by an argument involving a recursive class-sign more complicated than Gödel's **r**(**v**), established an 'Unprovability' Theorem for P (and for systems of similar character) which required as a condition only that P is 'consistent'.

A principal aim of Hilbert and his school had been to establish the 'consistency' of a calculus capable of being interpreted as expressing arithmetic, and thus to prove the consistency of a deductive system of arithmetic. To them the second great theorem contained in this paper was even more of a shock than the 'Unprovability' Theorem. For this second theorem proves the undecidability within P of a formula expressing the 'consistency' of P, thus showing that the 'consistency' of P, if P is 'consistent', cannot be established by a 'proof' within P, i.e. a 'proof' starting with only the 'axioms' of P and using only P's 'rules of inference'. [If P is 'inconsistent', of course both P's 'consistency' and P's 'inconsistency' can be 'proved' within P.]

THE 'UNPROVABILITY'-OF-'CONSISTENCY' THEOREM FOR P. Gödel proves this theorem (his Proposition XI: p. 70) in a general form, corresponding to that of his Proposition VI, which is concerned with 'deductions' as well as 'proofs' within P. As with Proposition VI I shall discuss Proposi-

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