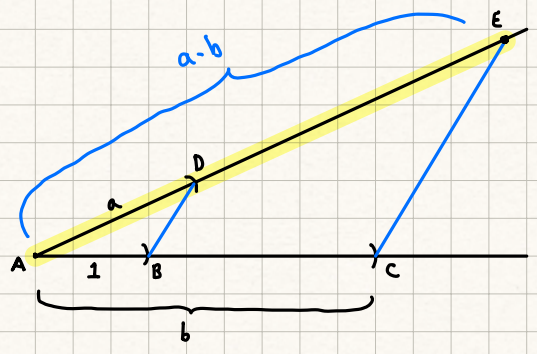
6. **Constructible Numbers** – build a number system from axioms of Euclidean Geometry.

Given: segment which is labelled "1."

C1: All natural numbers are constructible by repeated addition.

C2: Given a and b, we can construct $a+b$ and $a-b$.

C3: Given a and b, we can construct $a \cdot b$.

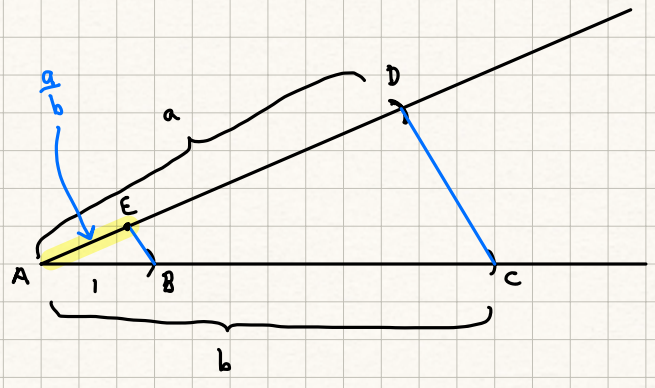


$\Delta ADB \sim \Delta AEC$

$$\frac{1}{b} = \frac{a}{x}$$

$$x = a \cdot b \quad (QED)$$

C4: Given a and b, we can construct $\frac{a}{b}$



$\Delta AEB \sim \Delta ADC$

$$\frac{1}{b} = \frac{x}{a}$$

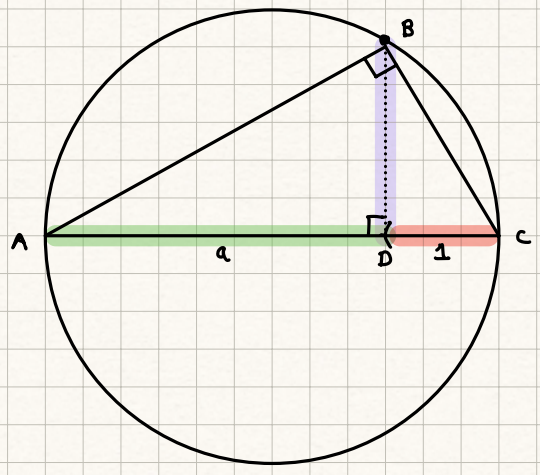
$$x = \frac{a}{b}$$

C5: All rational numbers are constructible.

(Side Note: Pythagoras and $\sqrt{2}$)

Question 1: Given a, can we construct $\sqrt{a}$?

C6: Given a, we can construct $\sqrt{a}$.



$\Delta ABC \sim \Delta ADB \sim \Delta BDC$

$$\frac{m\overline{AD}}{m\overline{DB}} = \frac{m\overline{BD}}{1}$$

$$\frac{a}{x} = \frac{x}{1}$$

$$a = x^2$$

$$\sqrt{a} = x \quad \text{(QED)}$$

Question – Can we construct $\sqrt[n]{a}$ for any $n \in \mathbb{Z}^+$, given a ?

- $a^{p/q} = \left(a^{\frac{1}{q}}\right)^p$ where $p, q \in \mathbb{Z}^+$
- Given any number b and any $m \in \mathbb{Z}^+$, we can construct b^m by repeated multiplication.
- If q were negative and we can construct $a^{\frac{1}{|q|}}$, then we can construct $a^{\frac{1}{q}}$ by division:

$$a^{\frac{1}{q}} = \frac{1}{a^{1/|q|}}.$$

- Main issue is, given a and $q \in \mathbb{Z}^+$, can we construct $a^{1/q}$?

7. **Abstract Algebra** – a study of the structure of number systems with a binary operation.