8. **Constructible Numbers Revisited** - (using Analytic Geometry)

- **Lemma 1:** If $q \in \mathbb{Q}$, then given a unit segment, q can be constructed.

- **Lemma 2:** One application of a straight-edge in $\mathbb{Q}^2$ will give a point w/ coordinates in $\mathbb{Q}^2 = \{(a,b) \mid a,b \in \mathbb{Q}\}$.

Pf: Let (a,b) and $(c,d) \in \mathbb{Q}^2$ determine line l_1 and (e,f) and $(g,h) \in \mathbb{Q}^2$ determine line l_2. Then, the equations for l_1 + l_2 are:

① l_1: $y = \frac{d-b}{c-a}(x-a) + b = m_1 x + b_1$, where $m_1, b_1 \in \mathbb{Q}$.

② l_2: $y = \frac{h-f}{g-e}(x-e) + f = m_2 x + b_2$, where $m_2, b_2 \in \mathbb{Q}$.

The intersection of l_1 and l_2 is given by the solution to:

$$m_1 x + b_1 = m_2 x + b_2$$

$$(m_1 - m_2)x = b_2 - b_1$$

$$x = \frac{b_2 - b_1}{m_1 - m_2} \in \mathbb{Q}.$$

$$y = m_1\left(\frac{b_2 - b_1}{m_1 - m_2}\right) + b_1 \in \mathbb{Q}.$$

Q.E.D.

- **Lemma 3:** A single application of a straight-edge and compass in $\mathbb{Q}^2$ will give a point with coordinates in some quadratic extension field of $\mathbb{Q}$.

Pf: Let line l w/ equation $y = mx + b$, $m, b \in \mathbb{Q}$ and circle C w/ equation $(x-h)^2 + (y-k)^2 = r^2$, where $h, k, r \in \mathbb{Q}$, be given. To find the intersection, we solve

$$(x-h)^2 + ((mx+b) - k)^2 = r^2$$

$$(x-h)^2 + (mx + b - k)^2 - r^2 = 0$$

The LHS has the form $ax^2 + bx + c$, where $a, b, c \in \mathbb{Q}$.

Then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\dfrac{b}{2a} \pm \dfrac{1}{2a}\sqrt{b^2-4ac} = p + q\sqrt{t}$,

where $p, q, t \in \mathbb{Q}$. So $x \in \mathbb{Q}(\sqrt{t})$. Since $y = mx + b = m(p + q\sqrt{t}) + b$

$= mq\sqrt{t} + (mp + b)$, where $mq,\ mp+b \in \mathbb{Q}$, then $y \in \mathbb{Q}(\sqrt{t})$.

QED

- **Lemma 4:** A single application of a compass in $\mathbb{Q}^2$ will result in a point whose coordinates lie in some quadratic extension field of $\mathbb{Q}$.

  Pf: Left to reader.

- **Lemma 5:** In $\mathbb{F}^2$, where $\mathbb{F}$ is any field, a single application of Euclidean Tools in $\mathbb{F}^2$ will give a point with coordinates in some quadratic extension field of $\mathbb{F}$.

  Pf: Follows directly from the proofs of Lemma 3 and 4, since no part of those proofs depended on the field being $\mathbb{Q}$ in particular.

- **Theorem:** Given a unit segment, a number $r \in \mathbb{R}$ is constructible iff r is in some nested quadratic extension field of $\mathbb{Q}$.

  Pf: Straight-forward application of previous lemmas.

  Ex:

$\sqrt{17}$

$3 + 5\sqrt{17}$

$(2 + 9\sqrt{17}) + (4 - 3\sqrt{17}) \cdot \sqrt{11 - 33\sqrt{17}}$

$$\left[(2 + 9\sqrt{17}) + (4 - 3\sqrt{17}) \cdot \sqrt{11 - 33\sqrt{17}}\right] +$$

$$\left[(9 - 7\sqrt{17}) + \left(11 + \tfrac{2}{3}\sqrt{17}\right) \cdot \sqrt{11 - 33\sqrt{17}}\right] \cdot$$

$$\sqrt{\left[\left(\tfrac{4}{7} - \tfrac{1}{5}\sqrt{17}\right) + (23 - 54\sqrt{17}) \cdot \sqrt{11 - 33\sqrt{17}}\right]}$$

$\sqrt[4]{7} = 0 + 1 \cdot \sqrt{0 + 1 \cdot \sqrt{7}}$

$\cancel{\sqrt[3]{7}}$ $\quad$ $\cancel{e}$

$\cancel{\sqrt[6]{7}}$ $\quad$ $\cancel{\pi}$