9. **Famous 3 Greek Problems** – Using only Euclidean Tools, it is impossible to:

1. Double the volume of a given cube.
2. Square a given circle.
3. Trisect an arbitrary angle.

① **Doubling the Cube** – Given a cube with side length x, it is impossible to construct a cube whose volume is twice that of the original.

Pf: Let a cube w/ edge length x be given. Then its volume is x^3.

BWOC, suppose we able to construct a cube with volume $2x^3$.

Then, the edge length of that cube would be $\sqrt[3]{2} \cdot x$. Dividing by x, we can construct $\sqrt[3]{2}$. $\lightning$ QED

② **Squaring the Circle** – Given a circle, it is impossible to construct a square w/ the same area as the given circle.

Pf: Left to you on the project.

③ **Trisecting an Angle** – It is impossible to develop a method that will trisect every angle.

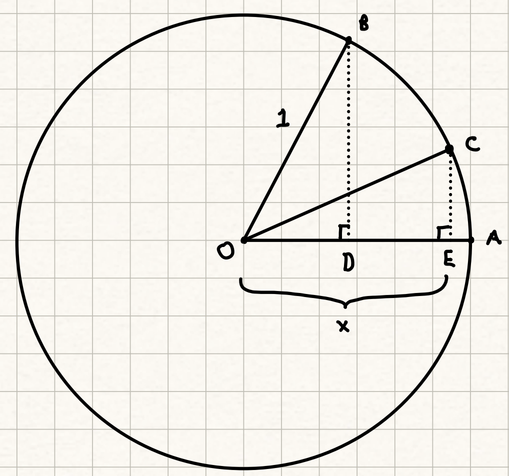
Lemma: $\cos x = 4\cos^3\left(\frac{x}{3}\right) - 3\cos\left(\frac{x}{3}\right)$.

Pf: By the angle addition formula,

$$
\begin{aligned}
\cos(3\theta) = \cos(\theta + 2\theta) &= \cos\theta\cos 2\theta - \sin\theta \sin 2\theta \\
&= \cos\theta\left(\cos^2\theta - \sin^2\theta\right) - \sin\theta \cdot 2\sin\theta\cos\theta \\
&= \cos^3\theta - \cos\theta\sin^2\theta - 2\sin^2\theta\cos\theta \\
&= \cos^3\theta - 3\cos\theta\sin^2\theta \\
&= \cos^3\theta - 3\cos\theta\left(1 - \cos^2\theta\right) \\
&= \cos^3\theta - 3\cos\theta + 3\cos^3\theta \\
\cos 3\theta &= 4\cos^3\theta - 3\cos\theta.
\end{aligned}
$$

Let $x = 3\theta$ to obtain $\cos x = 4\cos^3\left(\frac{x}{3}\right) - 3\cos\left(\frac{x}{3}\right)$. $\square$

Proof of Theorem – It suffices to show the existence of a single angle which cannot be trisected using only Euclidean Tools.



$m\measuredangle AOB = 60°$

Suppose BWOC that we are able to construct $\measuredangle AOC \ni: m\measuredangle AOC = \frac{1}{3} m\measuredangle AOB$.

Construct segments $\overline{BD}$ and $\overline{CE}$ perpendicular to $\overline{OA}$ as shown. Since $m\overline{OB} = m\overline{OC} = 1$, then $m\overline{OD} = \cos(m\measuredangle AOB)$ and $m\overline{OE} = \cos(m\measuredangle AOC)$.

Let $\theta = m\measuredangle AOB$ (so that $\frac{\theta}{3} = m\measuredangle AOC$). By the Lemma,

$$\cos\theta = 4\cos^3\left(\frac{\theta}{3}\right) - 3\cos\left(\frac{\theta}{3}\right)$$

Let $x = \cos\left(\frac{\theta}{3}\right)$; then

$$\frac{1}{2} = 4x^3 - 3x$$

$$1 = 8x^3 - 6x$$

$$8x^3 - 6x - 1 = 0$$

Since $m\overline{OE} = \cos\left(\frac{\theta}{3}\right) = x$, we have constructed a segment whose length is a solution to $8x^3 - 6x - 1 = 0$. It remains to show that all roots of $8x^3 - 6x - 1$ are non-constructible numbers.

By the Rational Root Theorem, if $\frac{p}{q}$ is a root of $8x^3 - 6x - 1$, then $p \mid 1$ and $q \mid 8$. Therefore, the only possible rational roots of $8x^3 - 6x - 1$ are $x = \pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{1}{4}$, and $\pm\frac{1}{8}$. It is easy to verify that none of these are roots. If a number $a + b\sqrt{c} \in \mathbb{Q}(\sqrt{c})$ is a root, then $a - b\sqrt{c}$ must also be a root (you will prove this). Also, since the roots of $8x^3 - 6x - 1$

must sum to the second leading term's coefficient (which is 0), if r is the third root (by Fundamental Theorem of Algebra):

$$(a + b\sqrt{c}) + (a - b\sqrt{c}) + r = 0$$

$$2a + r = 0$$

$$r = -2a \in \mathbb{Q}$$

↯ since $8x^3 - 6x - 1$ has no rational roots.