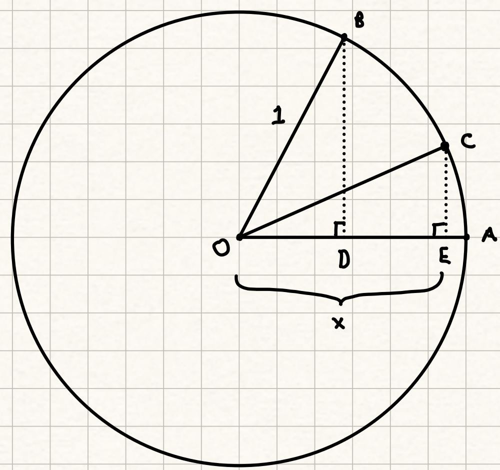
Proof of Theorem – It suffices to show the existence of a single angle which cannot be trisected using only Euclidean Tools.



$m\measuredangle AOB = 60^\circ$

Suppose BWOC that we are able to construct $\measuredangle AOC \ni: m\measuredangle AOC = \frac{1}{3} m\measuredangle AOB$.

Construct segments $\overline{BD}$ and $\overline{CE}$ perpendicular to $\overline{OA}$ as shown. Since $m\overline{OB} = m\overline{OC} = 1$, then $m\overline{OD} = \cos(m\measuredangle AOB)$ and $m\overline{OE} = \cos(m\measuredangle AOC)$.

Let $\theta = m\measuredangle AOB$ (so that $\frac{\theta}{3} = m\measuredangle AOC$). By the Lemma,

$$\cos\theta = 4\cos^3\left(\frac{\theta}{3}\right) - 3\cos\left(\frac{\theta}{3}\right)$$

Let $x = \cos\left(\frac{\theta}{3}\right)$; then

$$\frac{1}{2} = 4x^3 - 3x$$

$$1 = 8x^3 - 6x$$

$$8x^3 - 6x - 1 = 0$$

Since $m\overline{OE} = \cos\left(\frac{\theta}{3}\right) = x$, we have constructed a segment whose length is a solution to $8x^3 - 6x - 1 = 0$. It remains to show that all roots of $8x^3 - 6x - 1$ are non-constructible numbers.

By the Rational Root Theorem, if $\frac{p}{q}$ is a root of $8x^3 - 6x - 1$, then $p \mid 1$ and $q \mid 8$. Therefore, the only possible rational roots of $8x^3 - 6x - 1$ are $x = \pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{1}{4}$, and $\pm\frac{1}{8}$. It is easy to verify that none of these are roots. If a number $a + b\sqrt{c} \in \mathbb{Q}(\sqrt{c})$ is a root, then $a - b\sqrt{c}$ must also be a root (you will prove this). Also, since the roots of $8x^3 - 6x - 1$

must sum to the second leading term's coefficient (which is 0), if r is the third root (by Fundamental Theorem of Algebra):

$$(a+b\sqrt{c}) + (a-b\sqrt{c}) + r = 0$$

$$2a + r = 0$$

$$r = -2a \in \mathbb{Q}$$

↯ since $8x^3 - 6x - 1$ has no rational roots.

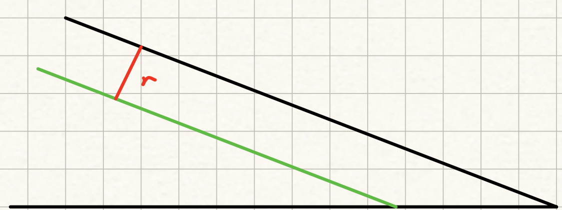
Suppose there exists a root in some nested quadratic extension field and let $\mathbb{F}$ be the smallest extension field that contains a root. Then $\mathbb{F} = \{a + b\sqrt{c} \mid a, b, c \in \mathbb{F}'\}$, where $\mathbb{F}'$ is the base field. The same proof shows that if $a + b\sqrt{c}$ is a root, then $a - b\sqrt{c}$ must also be a root and therefore the third root must be $r = -2a \in \mathbb{F}'$. ↯ since $\mathbb{F}$ is the smallest extension field containing a root.

Therefore, no elements of any nested quadratic extension field are roots of $8x^3 - 6x - 1$, so all roots are non-constructible numbers and hence our assumption that we trisected the angle must be false.

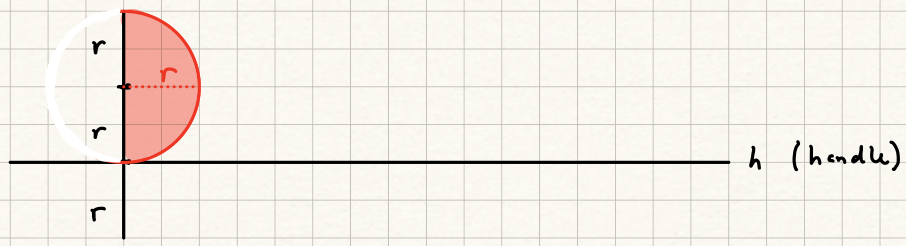
QED

10. **Tomahawk Method of Angle Trisection** – a Tomahawk is a second-order Euclidean Tool (can be constructed using Euclidean tools).

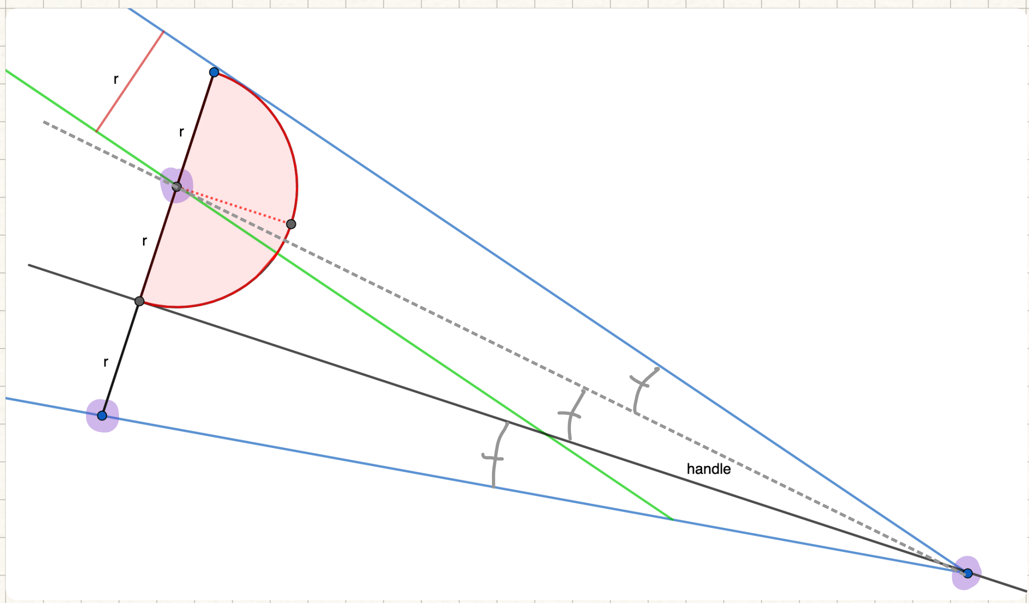
Step 1: Prepare the angle, given a length r.



Step 2: Construct Tomahawk



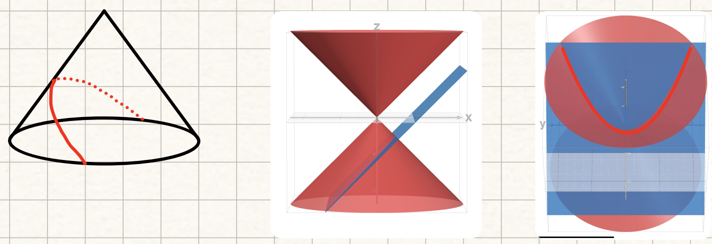
Step 3:



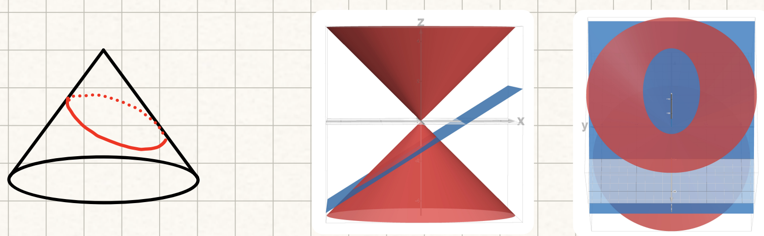
Conic Sections

1. **Definition from a Cone** – obtained by slicing a (double) cone w/ a plane.

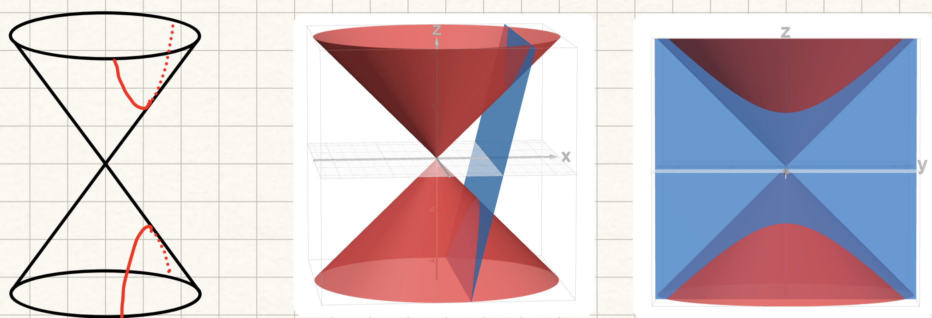
- **Parabola** – slice a cone w/ a plane at an angle equal to the base angle of the cone.



- **Ellipse** – slicing a cone at an angle less than base angle of the cone.

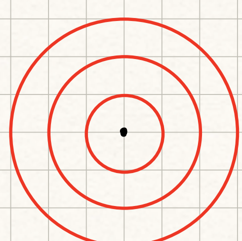


- **Hyperbola** – slicing a cone at an angle greater than the base angle of cone.



2. **Definitions using Loci** – a locus of points is a collection of points satisfying a geometric condition.

Ex: Locus of points equidistant from a given point.



several different loci