A Brief Overview of Non-Euclidean Geometries

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The field of Non-Euclidean geometries is relatively new (i.e. developed in the last few centuries) and arose out of efforts to prove or disprove the independence of Euclid's fifth postulate (the parallel postulate). This became part of a larger effort, spearheaded by Hilbert, to completely axiomatize all of mathematics in general and all of geometry in particular. When placed under close scrutiny, Euclid's geometry is far from a flawless example of a rigorous axiomatic system. I will briefly outline the flaws that were uncovered in our modern effort to make Euclidean geometry rigorous, and then I will outline the various forms of Non-Euclidean geometry that have been developed.[1]

1 Flaws in Euclid's System of Geometry

Hilbert completely and rigorously axiomatized Euclid's geometry by defining 16 axioms, divided into four categories (Greenburg, pp. 70-102). His systematization of Euclidean Geometry is not the only possible one, nor is it the first, but it is widely regarded as the most intuitive and the most like Euclid's in spirit:

1. **Betweenness (4):** In Euclidean geometry, "between" is an undefined term, and he makes several assumptions about how points and lines that are "between" others do and do not separate lines and planes. Hilbert made the implicit assumptions of Euclid explicit in four axioms.
2. **Congruence (6):** The axioms not only refine the assumption about congruence that Euclid assumed but did not state, it also restates some of Euclid's five "common notions" as axioms (in particular, his notion about transitivity). Surprisingly, Hilbert found it necessary to *postulate* the SAS result for congruent triangles.
3. **Continuity (5):** Euclid makes some assumptions regarding existence in his proofs. For example, in his proof of Proposition 1 in his first book of *Elements*, he clearly assumes that two circular arcs which overlap intersect each other at a point, but there is nothing in his axioms to suggest that such a point must exist. Such concepts are so consistent with our experience that we can easily fail to recognize them as assumptions.
4. **Parallelism (1):** Here, Hilbert assumes Playfair's statement of the parallel postulate, namely that given a line and a point not on that line, there exists exactly one line parallel to the given line and through the given point.

2 Classifying Geometries According to the Parallel Postulate

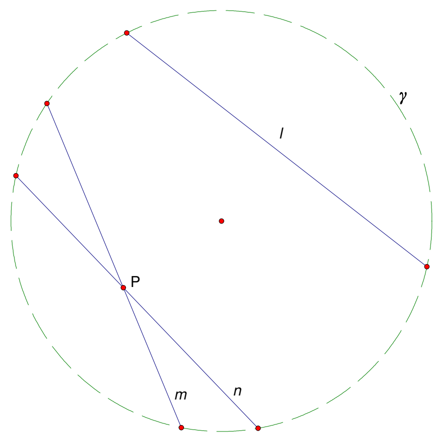
Because the field is so vast and the results both esoteric (i.e. understandable only to a limited group of experts) and complex, it is difficult to give an overview of this field. My attempt to do so will center around how each geometry treats Euclid's Parallel Postulate. (In the next section, I will organize the geometries in a slightly different way: according to how they treat the other classes of Hilbert's Axioms as stated below.) If we consider Playfair's version of Euclid's Parallel Postulate, three ways of treating this postulate become apparent: (1) accept it, (2) replace it by one of two possible negations, or (3) ignore it all together. Thus, four possible alternative geometries can be envisioned:

1. **Planar Geometry:** If we accept the parallel postulate, we of course obtain classic Euclidean Geometry. Since this geometry is most naturally rendered on a plane, it is called Planar Geometry. This completes our terminology, for the other two geometries are Elliptic and Hyperbolic, and a Plane is the intermediate form between an Ellipsoid and a Hyperboloid.
2. **Hyperbolic Geometry:** One of the two possible negations of PP is the following one, first proposed by Lobachevsky in the early 1800's: given a line and a point not on the line, there exist *more than one* line parallel to the given line and through the given point. This results in *Hyperbolic* geometry. In an attempt to

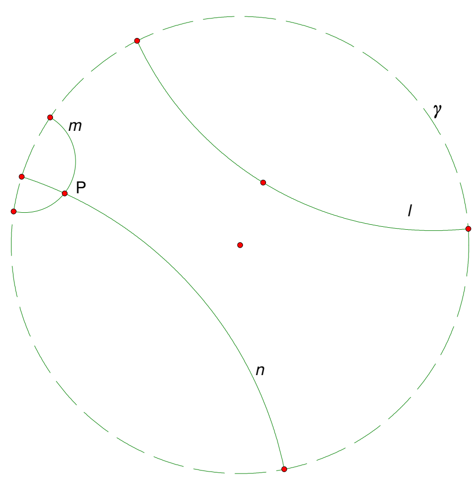
[1]The majority of the material in this exposition has been taken from *Euclidean and Non-Euclidean Geometries: Development and History, 3rd Edition*, by Marvin Jay Geenberg and *Introduction to Geometry, 2nd Edition*, by H.S.M. Coxeter, who is perhaps the premier geometer in the world. Another dependable site for information on geometry is The Geometry Center at the University of Minnesota, http://www.geom.uiuc.edu.

prove that such a geometry is consistent, several models were developed, and their structures explain the name given to the geometry. The first three models are continuous models within standard Euclidean geometry, which the last model is one that can be developed from alternative axioms and allows for a geometry that involves only a finite number of points and lines.

(a) Klein-Beltrami—In this model, a fixed circle, γ, in the Euclidean plane is considered to be the "plane," and open chords (cords of the circle, without their two endpoints) are considered to be "lines." The problems with this model are that the concepts of congruence and angle are very complex and unintuitive. However, it is clear that one can create more than one line parallel to a given line and through a given point. Restricting our attention to the circle γ guarantees the many chords which are not parallel in the Euclidean sense will not intersect in the Klein-Beltrami sense, since the Euclidean lines on which those chords lie will intersect outside of the circle. We use open chords in order to make the "lines" homeomorphic to Euclidean lines (i.e. we can make them have infinite length, if we adjust our measure correctly, because they don't have an "end"). In the figure below, lines m and n both pass through P but do not intersect l and are therefore "parallel" to l.

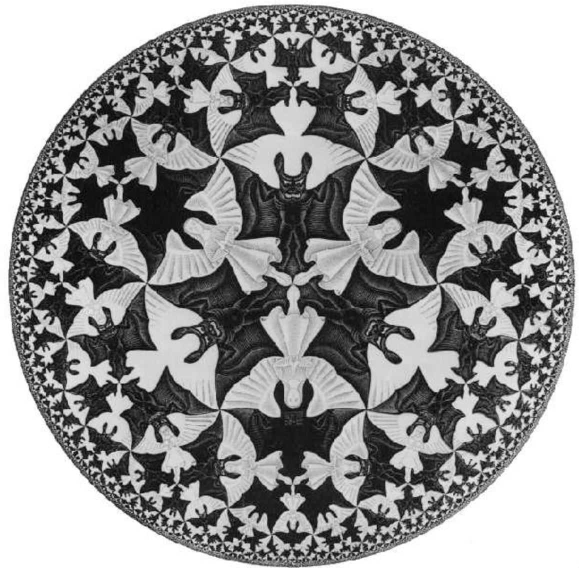


(b) Poincaré—This model begins with the same fixed circle, γ, and "lines" are open segments of circles which are orthogonal to γ. See the figure below. The advantage of this model is that the concepts of angle are the same as in standard Euclidean geometry, but the notion of distance is still somewhat different, because the closer one gets to γ, the more "distance" is diminished. In the figure below, again, lines m and n do not intersect l but pass through P.

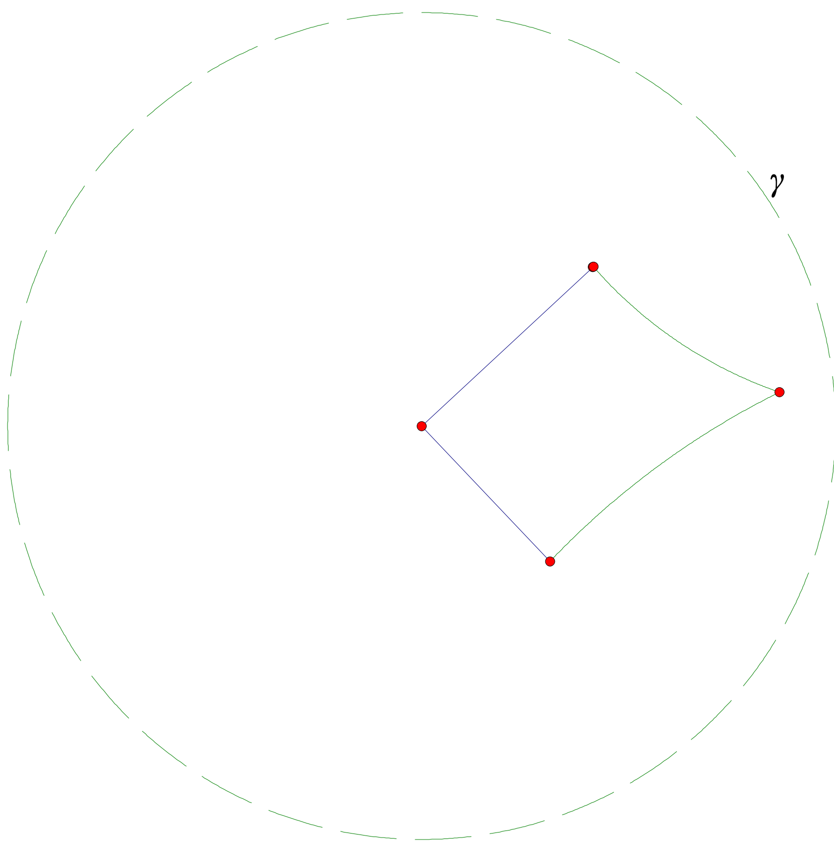


M.C. Escher made great use of the Poincaré model of the plane in some of his artwork. In the tiling in the picture below[2], each of the "angels" and "devils" is the same "size" according to the measuring scheme used in Poincaré's model.

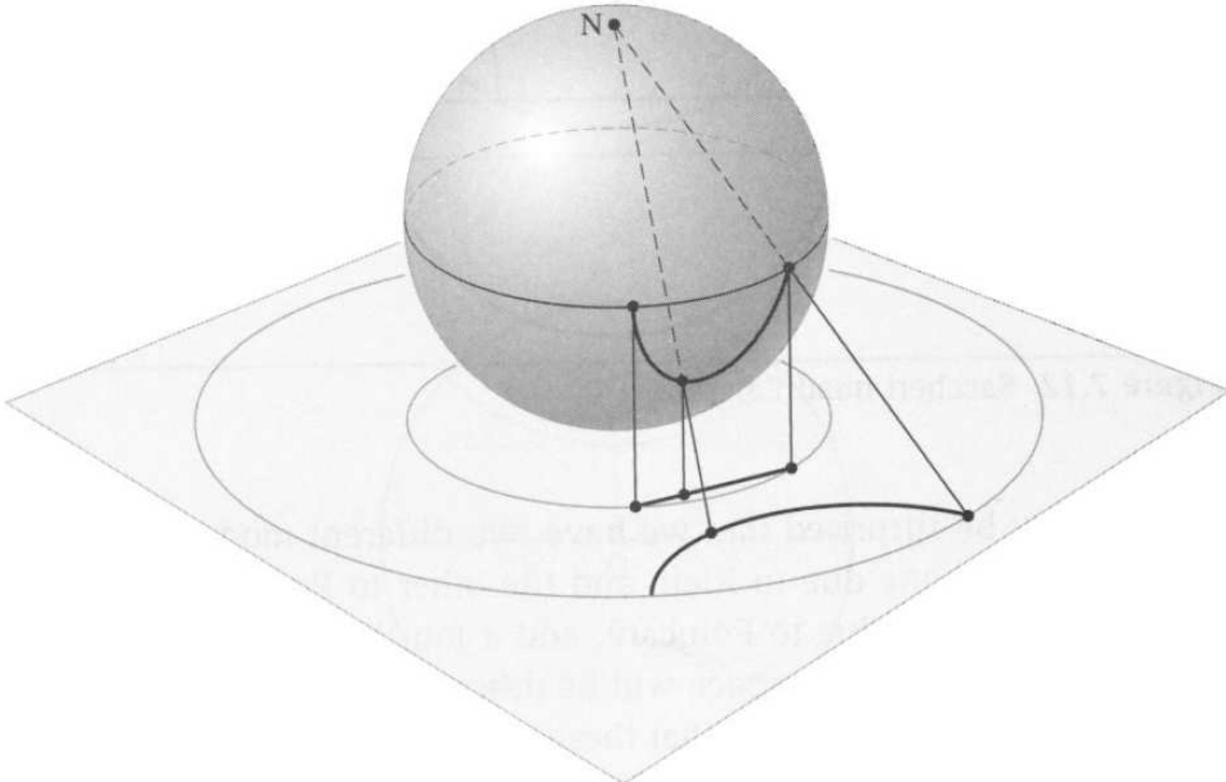
[2] *Circle Limit 4: Heaven and Hell* by M.C. Escher, 1960.



Another surprising result in this model of geometry is that the sum of the angles of a triangle is no longer 180° but is always less than 180°. This result makes possible such objects as a quadrilateral (called a "Lambert Quadrilateral") with 3 right angles and a fourth acute angle shown in the figure below.



Interestingly enough, the Klein and Poincarè Models are isomorphic (i.e. there is a bijective mapping between the two which preserves the concepts of incidence, betweenness, and congruence). The figure below shows this congruence. The Klein Model is represented by the smaller circle of radius 1 on the plane, and the Poincarè Model is represented by the larger circle of radius 2. Yet, both will map onto the lower hemisphere of a sphere of radius one centered at $(x, y, z) = (0, 0, 1)$ in an identical fashion. The Klein Model is projected vertically onto the sphere, whereas the Poincarè Model is projected stereographically, in which each point, P, on the plane is mapped to the intersection of the sphere with the line segment formed by P and N, the "north pole" of the sphere. One can prove that any "line" of in the Klein Model (i.e. an open chord within the circle of radius 1 lying on the xy-plane) maps onto a circular arc on the sphere that is exactly vertical and therefore orthogonal to the equator of the sphere. Since the stereographic projection is conformal (i.e it preserves angles), and since the circular arc on the sphere is orthogonal to the equator, it maps onto a circular arc on the xy-plane that is orthogonal to the circle of radius 2 (to which the equator of the sphere maps). Not only does this mapping establish a one-to-one correspondence between the points and lines of both models, but it preserves incidence, betweenness, and congruence.[3]



(c) Minkowski—This model envisions the "plane" as the top half of a hyperboloid of two sheets and the "lines" are the intersection of this sheet any plane passing through the center of the hyperboloid. An equivalence between this model and the Klein-Beltrami model is easy to see, as indicated in the figure below. If we project the "lines" in the Minkowski model onto a plane tangent to the top sheet of the

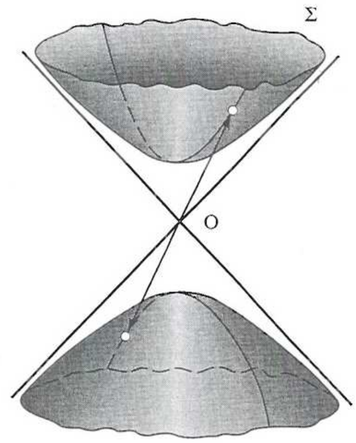
[3]Note that it is *not* true that any circle on the xy-plane will map onto an arc on the sphere that is exactly vertical, and therefore orthogonal to the xy-plane. However, the circular arcs that are orthogonal to the Poincarè Circle of radius 2 on the xy-plane will have this property. To see this, first note that, since the entire diagram is radially symmetric from the z-axis, it will suffice to show that *some* circle orthogonal to the Poincarè Circle of radius 2 in the xy-plane will map onto an arc on the sphere that is exactly vertical.

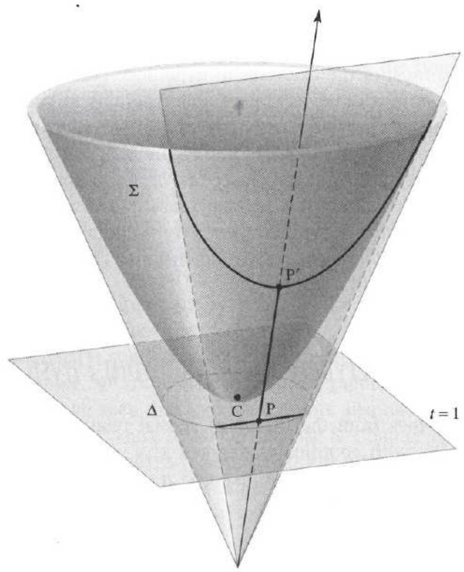
We will choose a circle whose center is located on the y-axis. First, we note that the line from the north pole, $(x, y, z) = (0, 0, 2)$ to an arbitrary point $(x, y, z) = (a, b, 0)$ in the xy-plane is given by $x = at,\ y = bt,\ c = 2 - 2t$ in parametric form (i.e. $t[a, b, -2] + [0, 0, 2]$). Substituting these values into the equation of the sphere, $x^2 + y^2 + (z - 1)^2 = 1$, and solving for t gives $t = \frac{4}{a^2+b^2+4}$. Subsituting this into the equation of our line gives the exact coordinates of the projection of our point onto the sphere: $(x, y, z) = \left(\frac{4a}{a^2+b^2+4}, \frac{4b}{a^2+b^2+4}, 2 - \frac{8}{a^2+b^2+4}\right)$.

The question is, what would a and b have to satisfy in order for the projection to have a constant value of y (i.e. for the projection to be exactly vertical)? We set $\frac{4b}{a^2+b^2+4} = k$ and solve, completing the square, to obtain $a^2 + \left(b - \frac{2}{k}\right)^2 = \frac{4}{k^2} - 4$. That is, a curve giving a completely vertical image on the sphere would have to lie on the circle $x^2 + \left(y - \frac{2}{k}\right)^2 = \frac{4}{k^2} - 4$. Notice that, since the radius must be positive, $0 < k < 1$, which also puts the center of this circle outside the Poincarè Circle of radius 2, which is correct.

The next question is, "is this circle orthogonal to the Poincarè Circle?" First, we solve for the intersection between it and $x^2 + y^2 = 4$ to obtain $y = 2k$ and $x = 2\sqrt{1 - k^2}$. Using Implicit Differentiation on both circles, we find that, for the Poincarè Circle, $y' = -\frac{x}{y}$ and, for the given circle, $y' = \frac{x}{\frac{2}{k} - y}$. At the relevant point of intersection, we see that the slope on the Poincarè Circle is $-\frac{\sqrt{1-k^2}}{k}$. The negative reciprocal of this is $\frac{k}{\sqrt{1-k^2}} \cdot \frac{\sqrt{1-k^2}}{\sqrt{1-k^2}} = \frac{k\sqrt{1-k^2}}{1-k^2}$, which is precisely what we obtain when we substitute the coordinates of the intersection into the formula for y' on the given circle. Hence, the circle that results in a completely vertical circular arc on the sphere is, in fact, orthogonal to the Poincarè Circle.

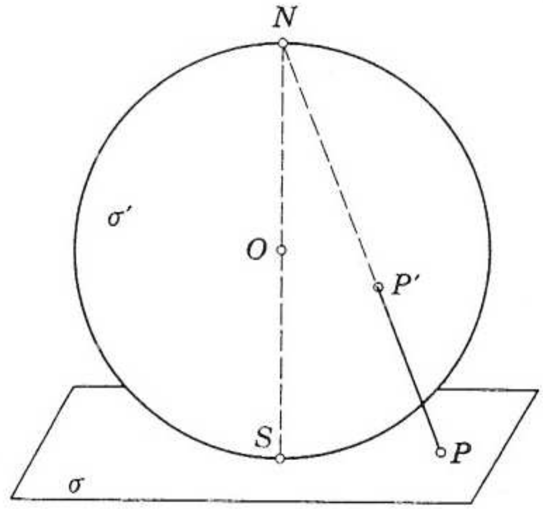
hyperboloid, the become exactly the chords to which the Klein-Beltrami model refer and, since the plane is really a "circle of infinite radius," the "chords" don't contain their endpoints.



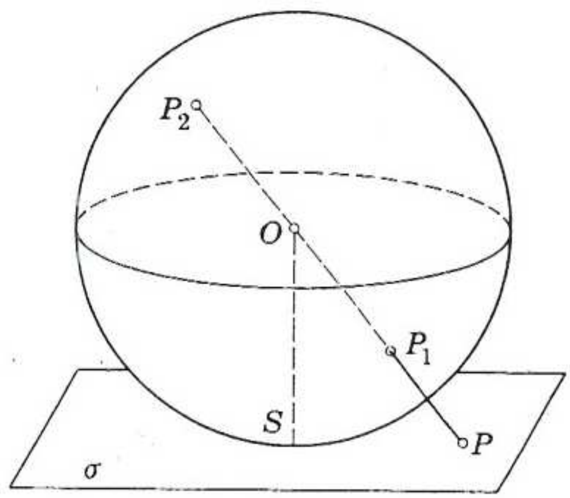


3. **Elliptic Geometry:** The other possible negation of PP is: given a line and a point not on the line, there exist *no* lines parallel to the given line and through the given point. Interestingly enough, the models developed from this assumption can also easily be developed from alternative axioms which allow for geometries with only a finite number of points and lines.

    (a) Projective Geometry—This geometry is created by mapping the Euclidean Plane onto a sphere as shown in the figure below. The origin becomes the "south pole" and infinity becomes the "north pole." Lines in the Euclidean plane become circles on the sphere which always pass through the north pole. Hence, all lines meet at the north pole, precluding any parallelism.



Alternatively, we can map the plane onto the lower hemisphere as shown below. In this case, all lines in the plane correspond to a great circle on the sphere. So, all lines parallel to a given line share the same point on the equator of the hemisphere. The equator is considered a "line at infinity," which consists of "all points at infinity."[4]



(b) Projective Finite Geometry—There is a geometry requiring only a finite number of points which nevertheless shares the properties of incidence and lack of parallelism outlined above. In particular, a projective plane consists of a set of lines, a set of points, and a relation between points and lines called incidence, having the following properties:

- Given any two distinct points, there is exactly one line incident with both of them.
- Given any two distinct lines, there is exactly one point incident with both of them.
- There are four points such that no line is incident with more than two of them.

The first two axioms guarantee that any two points determine a line (a Euclidean property) and that any two lines intersect (i.e. there are no parallel lines). The third condition precludes degenerate cases (such as a single point, or a single line, or a single line with multiple points on it), which are uninteresting. Notice that the first two axioms also establish a symmetry between the points and lines in this geometry, making them interchangeable in a sense. This property is called *duality*, and it means that any theorem that is proven immediately gives rise to another theorem that results from interchanging the roles of point and line in the original theorem. This geometry also contains many nice results regarding conics and transformations.

[4]One technicality is that to make everything work smoothly, we need to identify the antipodal points on the sphere; i.e. we consider any two antipodal points to be the same point. This idea is due to Kepler and Desargues.

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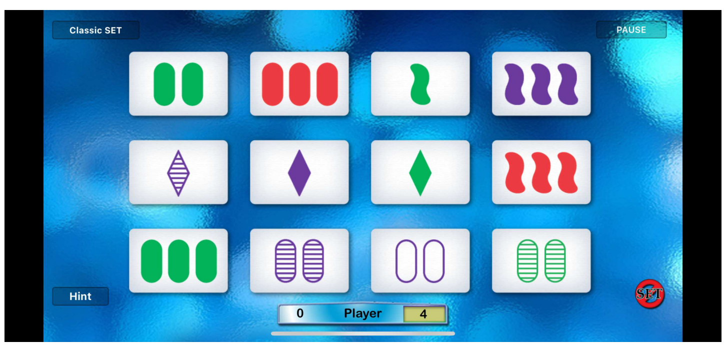
These are a type of minimal geometry, because they lack conventional notions such as distance and continuity. The advantage of this is that they help us to establish the minimum number of assumptions necessary to obtain certain geometric properties, since in larger geometries it is not always clear which notions are truly necessary to produce certain properties.

One tool that is used to help us visualize projective planes is vector spaces, and if the projective plane is finite, vector spaces over a finite field (i.e. 2-D vectors whose entries are numbers from that finite field). From Abstract Algebra, we know that finite fields can have $q = p^n$ elements, where p is a prime number and n is a positive integer. Such a geometry will have $q^2 + q + 1$ lines (and therefore $q^2 + q + 1$ points) with $q + 1$ points on each line (and therefore $q + 1$ lines through each point).

Each projective plane can give rise to affine planes, which are obtained by deleting exacty one line from the projective plane, along with every point that is on that line. One can prove that doing this creates a finite geometry in which Playfair's version of the Euclidean parallel postulate holds: given a line l and a point P not on that line, there exists one and only one line through the given point which does not intersect the given line.

It may seem like such geometries are mere mathematical curiosities, created for the amusement of math geeks who have nothing better to do with their time. However, these geometries are surprisingly useful in modeling various behaviors involving patters of finite numbers of object. One popular card game is SET, in which each card contains a particular number (1, 2, or 3) of a certain kind of shape (one of three possibilities), that is a particular color (red, green, or blue) and has a certain pattern (hollow, solid, or striped). A SET is obtained by finding three cards that, for each of the 4 characteristics, are either all the same or all different. For example, if each card has one oval, but one oval is green, one blue, and one red, and one oval is solid, one hollow, and one striped, that is a SET. The three cards agree on the number of shapes (1), the kind of shape (1), but differ in the color and the pattern. A student of a colleague of mine proved that this game can be modeled by a 2-D affine plane with 9 points and lines. Each card corresponds to a point in this geometry, and three cards form a SET iff the points representing them are colinear. Can you find a SET in the cards pictured below?



Additionally, this geometry shows up when examining something called octonians, which are a group structure used in analyzing results from quantum mechanics.

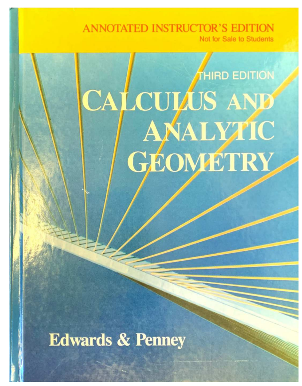
4. **Absolute Geometry:** First proposed by Bolyai around the mid-1800's, it is also called *Neutral Geometry*, and it results from asking what type of geometry is obtained if we invoke only Euclid's first four postulates and make no claim about parallelism at all. Such a geometry in some sense will be strange at first, because *parallelism* won't even be a notion. However, the advantage of developing this geometry is that whatever is true in this core of geometry will also be true in Euclidean (Planar), Elliptic, and Hyperbolic geometries, since all of those geometries share Euclid's first four postulates. In a sense, because it relies on fewer axioms, this geometry is a *broader, more inclusive geometry* of which the other three are specific examples.

3 Alternative Classification of Geometries

1. **Synthetic vs. Analytic:** This is such a vast subject that there are many ways to classify geometries, many of which overlap. One way is to distinguish between *synthetic* and *analytic* geometries.

(a) *Synthetic Geometry:* is the construction-based geometry that the ancient Greeks developed. In this geometry, construction tools are used whose function depends on the axioms of the geometry. Therefore, any construction that is made with these tools will not exist apart from a proof from the axioms that prove what has been constructed. In Euclid's Geometry, there can be made allowances for measurements, but not equations that describe objects.

(b) *Analytic Geometry:* was not developed until the time of Rene' Descartes, whose photo is shown below. He was the first to overlay 2-D space with a coordinate grid, in which each point on the plane could be identified by an ordered pair of numbers (x, y). We take this for granted today, but it was one of the most significant developments in mathematics, because it allowed for algebraic equations to be written that describe geometric objects (such as lines, circles, parabolas, hyperbolas, and ellipses). This allowed the field of algebra, which had developed in a way almost completely divorced from geometry. In fact, The word "algebra" is derived from an Arabic word in a treatise written in the year 830 by the medieval Persian mathematician, Muhammad ibn Mūsā al-Khwārizmī, whose Arabic title can be translated as *The Compendious Book on Calculation by Completion and Balancing.* The reason this unification between geometry and algebra was so momentous is that it allowed the entire arsenal of algebra to be used in solving geometry problems. This unified discipline is known as analytic geometry, and it lies at the basis of Calculus (as can be seen in the textbook pictured below).



2. **Hilbert's Classification:** Another way to classify geometries is according to how they include, exclude, or alter the notions identified by Hilbert: Betweenness, Congruence, Incidence, and Parallelism. We have already classified geometries according to the latter criteria. Below, we will consider additional classifications of the geometries discussed above as well as some new types of geometries not considered above.

(a) *Incidence Geometry:* Just as Absolute Geometry ignores the notion of parallelism, resulting in a more basic and therefore more broad geometry that includes Planar, Elliptic, and Hyperbolic geometries, Incidence Geometry simply identifies a property of incidence, or continuity, but ignores completely any concepts of congruence, betweennness. Parallelism flows naturally out of the notion of incidence, but there are no axioms concerning parallelism in incidence geometry. It is important as a super-geometry whose results apply to any geometry specifying incidence.

(b) *Affine Geometry:* This geometry is developed from a set of axioms, and it includes Euclid's parallel postulate but ignores notions of angle and length. There are many ways to envision Affine Geometry, but perhaps the most intuitive is as the geometry of a collection of displacement vectors in a vector space; hence, Linear Algebra is the natural language to use when expressing structures in an affine space. Affine Geometry can be considered to be a special case of Incidence Geometry.

(c) *Projective Geometry:* Projective Geometry can also be developed as a special case of Incidence Geometry. To my knowledge, there is no finite, hyperbolic geometry that has been developed as a special case of Incidence Geometry.

(d) *Ordered Geometry:* This geometry was initiated by Bertrand Russell and is developed from a set of axioms specifying properties of betweenness. It is important as a super-geometry whose results apply to any geometry that includes a notion of betweenness (or order, or intermediacy). Affine geometry is a special case of an ordered geometry. However, in Projective Geometry, the notion of betweenness is ambiguous: consider a line in projective space mapped onto a sphere with the north pole representing infinity; given any three points on any line passing through the circle, it is impossible to identify one point as lying between the other two in a unique way. Hence, Projective Geometry is not a special case of and Ordered Geometry.

3. **Many Other Classifications:** We have not begun to exhaust the ways in which one can classify the different geometries. Another way is to look at transformations of various types (rotations, dilations, reflections, inversions, etc.) and determine which transformations leave various properties fixed. Depending on which properties are maintained under which transformations, we can classify various geometries. In short, this is a very rich field that has very deep links to both Group Theory and Complex Analysis, as well as Combinatorics and Number Theory (in particular, the finite geometries have deep ties to the latter two mathematical subdisciplines).

4 Fractal Geometry

Fractal Geometry is not developed from Euclidean Geometry by varying the fundamental axioms. Instead, Fractal Geometry exists within Euclidean Geometry but includes analytical tools that Euclidean Geometry lacks. For example, in standard Euclidean (Analytic) Geometry, objects are classified as 0-dimensional, 1-dimensional, 2-dimensional, etc. In other words, dimensions are whole numbers. Fractal Geometry allows for dimensions which are fractional, even irrational, and hence challenges us to rethink the meaning and uses of the concept of dimension. For example, the geometric object below on the left, a Sierpinski Triangle (the recursion is infinite), exists in Euclidean Geometry but escapes precise description using only Euclidean Analytic Tools. The dimension of this object can be calculated as $\frac{\ln 3}{\ln 2}$. Furthermore, this dimension has practical as well as theoretical uses. Fractal objects are infinitely complex and can be quite elaborate (as shown in the figure below at the right), exist in a wide variety of physical systems, and are characterized by their self-similarity.