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TEN THINGS TO CONSIDER WHEN TEACHING PROOF

What I wish I had known about teaching proof before I taught geometry

As I sat in a high school geometry class and observed a beginning teacher, Matt (a pseudonym), teaching proof for the first time, I was reminded of my own experiences in teaching formal proof to secondary school students. Matt seemed to struggle with some of the same challenges I encountered when I began teaching proof. For example, he seemed uncertain about which definitions and theorems would appear consistently throughout the course and thus should be emphasized. Such uncertainties are difficult to resolve until "curricular knowledge" (Shulman 1986) is acquired either through the experience of using particular curriculum materials over time or through other capacity-building support. Because of the building nature of Euclidean geometry, knowing which mathematical ideas are central to "doing proofs" is especially important. Acknowledging that geometry and proof are not synonymous, I write about teaching proof in the context of the geometry course, because in the United States, proof is typically addressed as a stand-alone topic in this course.

During my eight years as a secondary school mathematics teacher, I found teaching formal proof in geometry one of the greatest challenges, so I decided to investigate the experience of learning to teach it. I did so by stepping into Matt's tenth-grade geometry classroom. To assist me in studying his practice, I videotaped lessons across the first three years (2005–2008) that Matt taught proof, and I conducted interviews to help me understand the choices he made in his teaching. In this article, I report what I learned through a combination of my observations in Matt's classroom, my own experiences as a teacher of geometry proof, and my reading of the literature on proof. More specifically, I explore ten things I wish I had known about teaching proof in geometry before I taught it.

I WISH I HAD KNOWN THAT ...

1

The research on the van Hiele levels is pertinent to proof. Although I began teaching secondary school with a master's degree in mathematics education, I was unaware of the literature on the van Hiele levels in geometry. Dina van Hiele-Geldof and Pierre van Hiele, two teacher-researchers greatly concerned about the difficulties their students encountered, believed that secondary school geometry involves relatively high-level thinking (Fuys, Geddes, and Tischler 1988). From their experiences, they came to believe that students had not had enough previous experiences at the prerequisite lower levels. As a result, their research focused on the levels of thinking in geometry and the role of the teacher in helping students move from one level to the next (Fuys, Geddes, and Tischler 1988). **Table 1** gives a summary of the van Hiele levels.

The van Hiele model of geometric thought can be used by teachers to guide instruction and assess students (Crowley 1987). Citing past research, Senk (1985) pointed out that "the van Hiele model, which posits the existence of discrete levels of geometric thought and ideas on how best to help students through the levels, has been used to explain why many students have difficulty with the higher-order cognitive processes, particularly proof, required for success in high school geometry" (p. 448). Thus, a greater awareness of these levels in conjunction with an understanding of what levels our students are working at can help us help our students. In addition, as I will stress later, students' lack of opportunities to engage in appropriate activities at the earlier levels points to a curricular issue in school mathematics. (To learn more about these levels and why they are important, see Crowley [1987] and Senk [1985].)

2

I should think about proof as a problem-solving activity.

I have come to believe that proof is difficult for many students because it is a problem-solving activity, one that cannot be proceduralized. In contrast to arithmetic and algebra (or at least many students' earlier mathematical experiences with these), proof is not an activity that can be done routinely. Although the two-column form provides a structure that helps scaffold proof for students, there is little a teacher can do—beyond telling students to write down the "givens"—to turn this activity into something less than problem solving. I am not saying that this is a bad thing. I only point out that if students have not had sufficient problem-solving experiences before the geometry course, they are likely to find doing proofs an unfamiliar, challenging, and frustrating experience. Again, this is a curricular issue as well.

Table 1

Summary of the van Hiele Levels and Their Characteristics

Levels	Characteristics
1 Visualization	The student identifies, names, compares, and operates on geometric figures (e.g., triangles, angles, intersecting or parallel lines) according to their appearance.
2 Analysis	The student analyzes figures in terms of their components and relationships among components and discovers properties and rules of a class of shapes empirically (e.g., by folding, measuring, using a grid or diagram).
3 Informal Deduction	The student logically interrelates previously discovered properties and rules by giving or following informal arguments.
4 Deduction	The student proves theorems deductively and establishes interrelationships among networks of theorems.
5 Rigor	The student establishes theorems in different postulational systems (i.e., non-Euclidean) and analyzes and compares these systems.

Source: Adapted from Crowley (1987) and Shaughnessy and Burger (1985)

3

I could play a more active role as an advocate for reasoning and proof throughout the curriculum.

If students are to be more successful with formal proof in the tenth-grade geometry course, they need to have more authentic mathematical experiences before taking geometry. As NCTM's *Principles and Standards for School Mathematics* notes, "Reasoning and proof cannot simply be taught in a single unit on logic, for example, or by 'doing proofs' in geometry.... Reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts" (NCTM 2000, p. 56).

I wish that I had made a conscious effort to make reasoning and proof a more central component in all my classes. When teaching algebra, teachers can point out the mathematical reasons that allow us to solve algebraic equations. The geometry course should not be the place where, for the first time, students learn about the reflexive property, for example. Ideally, such properties and their names will be used when appropriate before then. Students should not suddenly, in a geometry course, be inundated with formal names for properties, concepts, or theorems they have been using for many years.

Another curricular issue is students' lack of experiences with geometry concepts at van Hiele's level 3 (informal deduction). In her analysis of grade-level expectations across forty-two states, Newton (forthcoming) found that these expectations at van Hiele's level 3 were absent from more than 40 percent of states' standards, and in the remaining states there were very few such expectations at this level. Newton concluded that, despite thirty years of research that supports the use of the van Hiele level model, few changes have been made in the K–8 curriculum to reflect knowledge gained from this research. If we are to bridge the gap between K–8 mathematics and the high school geometry curricula, those who make curriculum decisions must make changes that reflect this knowledge. As Newton observes, *Mathematics education researchers* need to continue to refine instructional and assessment models for the levels; *state standards writers* need to use van Hiele's levels in creating consistent, supportive geometry learning trajectories when revising the documents; *curriculum developers* need to develop textbooks that better reflect the necessary geometric reasoning skills in coherent trajectories; and *teachers* need to be supported in recognizing and facilitating students' progression through the levels. (Newton, forthcoming, emphasis added)

Only this type of systemic approach, Newton notes, will adequately prepare students for proof in the geometry course.

4 *I should be more explicit about the purpose of proof.*

One category of proof studies found in the literature is related to philosophical issues about the definition and the role of proof. Among the various roles of proof that are cited, the most commonly discussed are *to convince* and *to explain*. Hersh (1993) noted that while a good mathematical proof can do both, at the high school or the undergraduate level, the primary role of proof is to explain. In contrast, I had always told my students that the role of proof was to convince, a perspective I now view as questionable.

This distinction between convincing and explaining is important because, in many cases, there is little doubt in students' minds that some particular proposition is true. For example, when provided with the base angles theorem and a corresponding diagram, most students do not need to be *convinced* that if two sides of a triangle are congruent, then the base angles are also congruent. However, students are often used to showing or explaining their work, so perhaps asking them to write a proof as a way of *explaining why* something is true would make more sense to them.

5 *I should be more explicit about the structure of proof.*

In most of the conventional textbooks I have used or analyzed, the authors seem to dive right into proof without paying much attention to its structure. More specifically, textbook authors do not generally explain that Euclid developed a system—a game, if you will. As with sports or other games, rules need to be established in order for people to play. Fawcett (1938) claimed that students should understand the following:

1. The place and significance of undefined concepts in proving any conclusion.
2. The necessity for clearly defined terms and their effect on the conclusion.
3. The necessity for assumptions or unproved propositions.
4. That no demonstration proves anything that is not implied by the assumptions. (p. 10)

Students should understand that without a starting point, we would be stuck in a endless cycle. Thus, Euclid based his system on postulates (statements that cannot be proved). In addition, to define a term, we need to have some undefined terms to work with (e.g., *point*). Other systems are possible, and other systems exist. These concepts were never made explicit to me, and I did not make them explicit to my students.

6 *I should take time to read and study NCTM's Standards publications while teaching geometry.*

NCTM's *Standards* publications should be required reading for preservice teachers. Unfortunately, however, they are not. Even if teachers read the *Standards* as undergraduates, they should revisit them when they are teaching. At a minimum, I recommend that teachers of geometry read the Reasoning and Proof Process Standard (NCTM 2000, pp. 56–63) and the Geometry Content Standard for grades 9–12 (pp. 308–18).

Unfortunately, one recommendation of NCTM's 1989 publication—the call to decrease attention on two-column proofs—was misinterpreted as a call to decrease proof, clearly not the intention of the authors. Rather, as NCTM makes clearer in its 2000 publication, the argument, not the form of the argument, is important: "The focus should be on producing logical arguments and presenting them effectively with careful explanation of the reasoning, rather than on the form of proof used (e.g., paragraph proof or two-column proof)" (p. 310).

Students in geometry class usually learn (by default) that all proof in elementary geometry is written in two columns (Usiskin 1980; Schoenfeld 1988). This is especially unfortunate since Matt, the teacher in my study, found that many of his students preferred the flow proof over the two-column proof (Cirillo 2008).

Further, as Usiskin (1980) pointed out nearly thirty years ago, some proofs should be removed from the curriculum. For example, Usiskin argued that some of the very early "rigorous" proofs of obvious statements should be treated informally. Matt followed this advice, replacing some of the more "semantical" proofs with other "more interesting proofs" to introduce his students to formal proof (see **fig. 1** for an example of what Matt called a "semantical" proof).

7 *Logic should be explicitly connected to Euclidean proof.*

One way to do this is through the law of syllogism, also known as the chain rule: $[(a \to b) \wedge (b \to c)] \to (a \to c)$. (See **fig. 2** for an example of how the law of syllogism could be demonstrated in a Euclidean proof.) Not all proofs, however, rely only on the law of syllogism. Using the flow proof to help make connections to the laws of logic—for example, the law of detachment, which may be stated as $[(p \to q) \wedge p] \to q)$—may be easier than using the two-column proof. As Matt said, the flow proof shows how all the different pieces "fit together" (Cirillo 2008, p. 158). A great activity that I have used to introduce my students to proof is the one described by Craine and Rubenstein (2000), who use an airline metaphor ("traveling toward proof") to support students' understanding of the deductive structure.

8 *Wait time is critical for creating space for student involvement.*

One aspect of Matt's teaching that changed over time was that he allowed more wait time. As Matt said, "These things … don't happen just like that. You have to give 'em time" (Cirillo 2008, pp. 231–32). The research on wait time is clear. Teachers interested in having students respond to questions with more than simple answers must give them time to think—at least three to five seconds of wait time in whole-class questioning and at least thirty seconds in discussion with a partner (Lee 2006).

After writing a problem on the board, Matt gave students time to think about the proof and to discuss it with a partner before the whole-class discussion. Over time, Matt's deliberate use of wait time led to increased student participation in the collective act of writing proofs. The use of wait time can help students come to believe that just because they do not see a path toward doing the proof immediately, this does not necessarily mean that they will not eventually be successful in writing the proof.

As Farrell (1987) pointed out, *doing* a proof and *writing* a proof are two different but important activities. The *doing* requires good problem-solving skills; the *writing* requires rigor and precision. However, the writing takes a back seat to the generating of ideas (Farrell 1987). Also, doing proofs is not typically a linear process, as the deductive style of presentation implies (Lakatos 1976). As Lakatos (1976) wrote, "Deductivist style hides the struggle, hides the adventure" (p. 142). We should think of proof as more of a "zig-zag path" between conjectures and refutations (Lampert 1992, p. 306).

9 *Students should conjecture, not just prove.*

If we are to engage students in more authentic mathematical practices, we must allow them to conjecture. After all, doing mathematics involves discovery, and conjecture (informed guessing) is a major pathway toward discovery (NCTM 2000). Thus, students at all levels "should learn to investigate their conjectures using concrete materials, calculators and other tools …" (NCTM 2000, p. 57).

An excellent example of a teacher's facilitation of conjecturing can be found in Cox (2004). In this article, Cox, an experienced geometry teacher, explained how she helped her students see how mathematicians use proof by having the students make conjectures based on observations, test the conjectures, and then work to justify the conjectures through proof.

Teachers can also facilitate conjecturing through observations by using dynamic software such as The Geometer's Sketchpad® or manipulatives such as compasses and Patty Paper®.

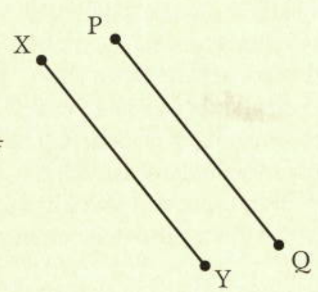
10 *It is important to teach proof, not theorems.*

Through his practice (unlike my own), Matt helped me see the importance of teaching the *proof process* rather than the *theorem content*. This important difference was noted by Fawcett (1938) seventy-one years ago: "[Observation of] actual classroom practice indicates that the major emphasis is placed on a body of theorems to be learned rather than on the *method* by which these theorems are established. The pupil feels that these theorems are important in themselves and in [his or her] earnest effort to "know" them [he or she] resorts to memorization" (p. 1).

One important call for change has been NCTM's (2000) recommendation that students of all ages should have experiences that facilitate an understanding of reasoning and proof as fundamental aspects of mathematics. However, research conducted more recently (e.g., Herbst and Brach 2006;

Given: $\overline{PQ} \cong \overline{XY}$

Prove: $\overline{XY} \cong \overline{PQ}$

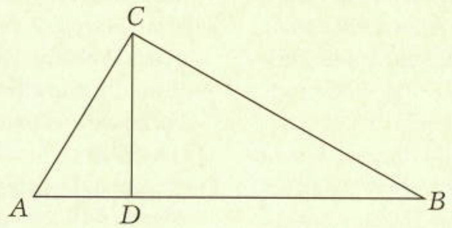


Statements	Reasons
1. $\overline{PQ} \cong \overline{XY}$	1. Given
2. $PQ = XY$	2. Definition of congruent segments
3. $XY = PQ$	3. Symmetric property of equality
4. $\overline{XY} \cong \overline{PQ}$	4. Definition of congruent segments

Source: Larson, Boswell, and Stiff (2001, p. 102)

Fig. 1 "Semantical" proofs should be given less attention.

Prove: In $\triangle ABC$, if $\overline{CD}$ is the altitude to $\overline{AB}$, then $m\angle CDA = 90°$.



A Logic Proof

Let A represent "$\overline{CD}$ is the altitude to $\overline{AB}$"

Let P represent "$\overline{CD} \perp \overline{AB}$"

Let R represent "$m\angle CDA$ is a right angle."

Let M represent "$m\angle CDA = 90°$."

	Statements	Reasons
1. A	$\overline{CD}$ is the altitude to $\overline{AB}$.	1. Given
2. $A \rightarrow P$	If $\overline{CD}$ is the altitude to $\overline{AB}$, then $\overline{CD} \perp \overline{AB}$.	2. Definition of altitude
3. P	$\overline{CD} \perp \overline{AB}$.	3. Law of detachment
4. $P \rightarrow R$	If $\overline{CD} \perp \overline{AB}$, then $m\angle CDA$ is a right angle.	4. Perpendicular lines intersect to form right angles.
5. R	$\angle CDA$ is a right angle	5. Law of detachment
6. $R \rightarrow M$	If $\angle CDA$ is a right angle, then $m\angle CDA = 90°$.	6. Definition of right angles
7. M	$m\angle CDA = 90°$	7. Law of detachment

A Geometry Proof

Statements	Reasons
1. $\overline{CD}$ is the altitude to $\overline{AB}$.	1. Given
2. $\overline{CD} \perp \overline{AB}$.	2. Definition of altitude
3. $m\angle CDA$ is a right angle	3. Perpendicular lines intersect to form right angles.
4. $m\angle CDA = 90°$	4. Definition of right angles

Note: Other examples that connect logic and geometry proofs can be found in Keenan and Dressler (1990)

Fig. 2 The law of syllogism connects logic to proof.

Herbst et al. 2009; Newton, forthcoming) indicates that changes proposed in NCTM's *Standards* publications related to the geometry curriculum and the teaching of proof have not occurred.

Part of the problem is the kinds of directives that both Matt and I were given by our department—for example, to "cover" the first six chapters of the textbook during the first semester of the course. This strategy emphasizes quantity over quality. I wish that I had known that teaching the *process* of proving is more important than covering a large quantity of theorems.

CONCLUSION

If we really are to heed the call to make school mathematics more authentic, we need to think carefully about how we teach proof. If, in fact, we are interested in having teachers teach proof rather than theorems, then we need to emphasize different kinds of curriculum objectives not related to "covering" particular chapters in a textbook. We also need to move away from using curriculum materials that present theorems in the "given/prove" format and move toward an emphasis on conjecturing. Mathematics supervisors and teachers need to develop more sophisticated curricular objectives related to what teaching the *process of proof* might look like.

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