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# Spadework Prior to Deduction in Geometry

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Consider the following excerpt from the transcript of a recent interview with a sophomore in high school who had successfully completed a year-long course in geometry. The interviewer asked, "Suppose you have a quadrilateral with both pairs of opposite sides congruent. Are the opposite sides parallel? How would you know for sure?" The student began to draw diagrams of quadrilaterals and even drew some auxiliary lines to search for congruent triangles but then paused for an extensive time ($I$ = interviewer, $S$ = student).

*I:* What are we trying to do here?

*S:* Make a proof.

*I:* Why do we want to do that?

*S:* I don't know, I could never figure it out.

*I:* Suppose that we wanted to know for sure that that [the statement] had to be true.

*S:* Oh no!

*I:* What would we do?

*S:* I told you I flunked proof! I think there's a theorem, but I can't remember....

*I:* I sort of get the impression they [theorems] are not your favorite things.

*S:* My teacher used to put super hard ones on there just so you'd get an F.

*I:* But you like math?

*S:* Yeah, but this geometry is not the same!

*I:* As what?

*S:* As math!

*I:* What do you like in math?

*S:* Algebra.

*I:* Why do you like algebra?

*S:* Because it's numbers. I'm not too logical.

*I:* Well, you've got a good start [on a proof during the interview].

*S:* Well, I'd go along and do a real great proof and then she'd say you need this step and this step, and I couldn't see why I needed those steps.

*I:* Did you do anything with areas and volumes?

*S:* Yeah, that's how I got my grade up to an A.

This student was one who, according to her teacher, had done very well in geometry. She was one of the few students we interviewed over a two-year period who actually recognized the need for a proof and made some progress toward constructing one. And yet the student obviously doesn't view herself as being successful in the geometry course. Geometry teachers know that this student's experience is not an isolated instance. Most students in high school geometry have a lot of difficulty with deduction and proof. They don't understand the role or meaning of an axiomatic system. Despite our best efforts to teach them, even the most capable algebra students may struggle and get through geometry by sheer willpower and memorization but with little understanding of the logical system we have been developing all year. Why do students have such a difficult time in geometry?

**Secondary school students should study geometry without proof for at least one-half year.**

For the past several years, we have been involved in a project that has investigated how elementary and secondary school students think about geometric concepts. The impetus for the project was a theory of "levels" of geometric thinking first set forth by two mathematics teachers from the Netherlands, Pierre van Hiele and Dina van Hiele-Geldof. In the project an activity-based interview was developed and administered over a two-year period to over seventy students in elementary and secondary school. The results of the interviews and the van Hiele theory have implications for the way geometry is taught in the schools and for the way students learn geometric concepts. In this article we shall describe the van Hiele levels and our interviewing activities, discuss some of the responses of the students to the activities, and then offer some suggestions for the teaching of geometry on the basis of the results.

### The van Hiele Levels

The van Hieles were mathematics teachers who met with the same difficulties that we all encounter in presenting formal deduction to geometry students. From classroom observations, the van Hieles felt that students passed through several levels of reasoning about geometric concepts. These levels have been described previously in the *Mathematics Teacher* (Hoffer 1981). We shall use the concept of a rectangle to illustrate the levels.

*Level 0—visualization.* At this level, a geometric figure is seen as a whole. No attention is given to its components. Descriptions are purely visual. If asked why he or she called a figure a rectangle, a student might reply, "Because it looks like a rectangle. It is like a window or a door." (These descriptions use a visual prototype.)

*Level 1—analysis.* Students at this level think of a rectangle as a collection of properties that it must have (necessary conditions). When asked why a figure is a rectangle, the student's response would be a litany of properties: "Opposite sides are parallel, opposite sides are congruent, opposite angles are equal, you have four right angles...."

*Level 2—informal deduction.* At this level, a student can select sufficient conditions from the "litany" just described to determine a rectangle. That is, the student orders properties logically and begins to appreciate the role of general definitions. Simple inferences can be made, and class inclusions are recognized (e.g., squares are rectangles).

**Activities involving informal geometry should be included at the junior high school level.**

*Level 3—formal deduction.* At this level the role of axioms, undefined terms, and theorems is fully understood, and original proofs can be constructed. Many high school courses presently approach the study of geometry at this level.

*Level 4—rigor.* Comparisons between different axiomatic systems can be made at this level. For example, what happens to geometry if we do not assume the parallel postulate? (This level is rarely encountered by high school students.)

### The Interviewing Tasks

One of the goals of our project was to develop some interviewing tasks to determine if the van Hiele levels are useful in describing students' preferred patterns of reasoning about geometric concepts. These tasks were designed so that they could be presented to students of all ages, from kindergartners to college students. Interviews were audiotaped in an informal session. For secondary school students, one of our concerns was how they differed in their responses to the activities before, during, and after a "traditional" course in high school geometry. We shall give a brief description of some of the tasks and then discuss some of the responses in more detail.

The tasks were (1) drawing; (2) identifying and defining; (3) sorting; (4) "what's my shape?" (an inference game); and (5) theorems, axioms, and proof.

*Drawing.* The student was asked to draw a triangle, draw one that was different in some way, then draw another one that was different from the first two, and so on. "How many different triangles can you draw? How are they different?"

*Identifying.* Given a sheet of geometric figures (see fig. 1), the student was asked to put an *S* on each square, an *R* on each rectangle, a *P* on each parallelogram, and a *B* on each rhombus. Then they were asked why they had made the markings they did and why they had not marked some of the figures. In the *defining* part of this activity, they were asked, "What would you tell someone to look for if they had to pick out all the rectangles on a sheet of figures? Could you make a shorter list? Is shape 2 a rectangle? Is shape 9 a parallelogram?" (In other words, the activity explored definitions and class inclusions. Similar questions were asked about squares, parallelograms, and rhombi.) An identifying activity was also done with triangular shapes (see fig. 2).

*Sorting.* A set of cutout figures were spread out on the table in front of the student, and then they were asked, "Put some of these together that are alike in some way. How are they alike? Put some together that are alike in a different way. How are they alike?" This line of questioning was continued as long as it seemed fruitful. Two sets of cutouts were used, triangles and quadrilaterals. Copies of the cutouts are shown in figure 4.

*"What's my shape?"* This game of inference was played with the students. They were told, "I'm going to show you a list of clues for a shape. I'll uncover the clues one at a time. When you have just enough clues to know for sure what shape it is, stop me. Otherwise, ask for another clue. Feel free to draw or to use any drawing apparatus on the table." When the students stopped us, we asked how they knew with certainty and whether another clue might change their minds. The list of clues for one of the shapes and the directions for the activity are given in table 1. (Uncover the clues one at a time to simulate our experiment. A solution was considered "correct" if the students identified the type of quadrilateral in the minimum number of clues necessary to determine shape.)

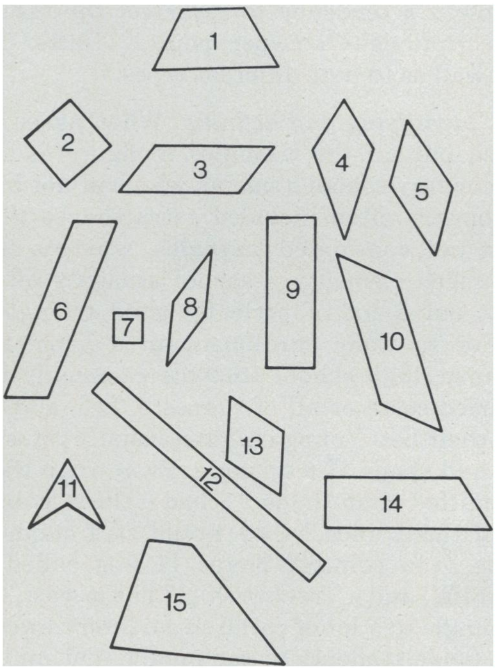

Fig. 1

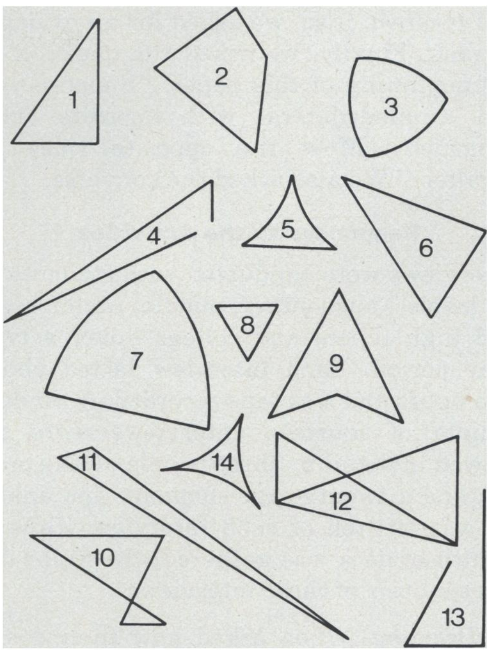

Fig. 2

TABLE 1

What's My Shape?

*Script* (Carefully give the directions below.)

1. I'm going to show you a sheet of paper with some clues about a certain shape. I will uncover the clues one at a time.
2. Stop me when you have just enough clues to know for sure what type of shape it is. Ask for another clue if you want one.
3. Make a drawing of the shape if you want to. Think aloud if you want to, and tell me what you are thinking about.

*Clues*

1. It is a closed figure with four straight sides.
2. It has two long sides and two short sides.
3. It has a right angle.
4. The two long sides are parallel.
5. It has two right angles.
6. The two long sides are not the same length.
7. The two short sides are not the same length.
8. The two short sides are not parallel.
9. The two long sides make right angles with one of the short sides.
10. It has only two right angles.

*Axioms, theorems, and proof.* This final activity was done only with the secondary school students. We asked them if they had ever heard of the words *axiom, postulate,* and *theorem.* If so, we asked for an example of each. Finally, we asked the question at the beginning of this paper, "Suppose you had a quadrilateral with opposite sides congruent. Must the opposite sides be parallel?" We also asked the converse.

### Responses to the Activities

Interviews were conducted with students at all levels—elementary, middle, junior high, and high school and college—over a two-year period. Each interview lasted about two hours and was tape-recorded. A random sample of fourteen interviews were reviewed in detail by three people on the project, and a twenty-page summary and analysis was written of each interview. Thus, a wealth of data was gathered; the following is a synopsis of these interviews.

*Drawing.* When asked how their drawings differed, many students replied that some of their triangles were "fatter" or "pointier" than others. Very young children usually felt that they could draw only a few triangles, perhaps three or four, and these frequently differed only in orientation (i.e., the direction they "pointed in"). The elementary and middle school students showed a preference for attending solely to the visual differences in the drawings. Pregeometry secondary school students preferred to describe differences in terms of the components of the shapes. For example, shapes differed because "the sides were longer." These students indicated that an infinite number of different triangles were possible. Students who were completing a year of geometry focused on "types" of triangles—right, isosceles, scalene, acute, and so forth. Students who had been out of geometry for a year or more showed a tendency to resurrect the visual descriptions—"sharper point," "fatter"—as well as to note different types.

**Students' concept of a triangle may include some shapes that are not triangles.**

*Identifying and defining.* When asked to pick out all the triangles in figure 2, the secondary school students who had not had geometry often included extra shapes that are not considered triangles, whereas the younger elementary school students often left out a lot of perfectly good triangles. Prior to their enrollment in a geometry course, high school students commonly included some or all of shapes 3, 7, 5, and 14 in their list. Young children usually refused to call shape 11 a triangle. Even when they admitted that shape 11 had "three points and three lines," they would still claim it was "too pointy." Shape 11 was called a "knife" and a "rocket ship," but it wasn't a triangle to a lot of children we interviewed.

Some students at the middle and junior high school levels felt that only shape 8 was a triangle. To them, it was the *perfect* triangle, and nothing else would do. One junior high school student picked out only shape 1 and said that the *other* triangle wasn't shown—the left triangle was shown, but the right triangle wasn't! This student felt that only "half" the triangle was given (see fig. 3).

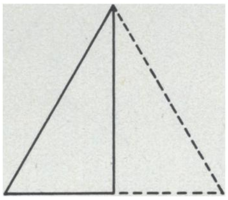

Fig. 3. The "right" part of the triangle was missing for this student (shape 1, fig. 1).

For figure 1, many students, including some who were taking geometry, turned the paper to see if shape 2 was a square or if shape 12 was a rectangle. The irrelevant attribute of orientation changed their responses, for when we turned the paper back again, they often said it was no longer a square but a diamond.

The rest of this activity was devoted to informal definitions of shapes. For example, the students were asked what they would tell a friend to look for so that she or he could pick out all the squares on a sheet of figures. The elementary school students just looked at us with a funny grin and said, "I'd tell them to pick out all the squares" or "look for the doors." High school students—before, during, and after geometry—usually gave a long litany of redundant properties: opposite sides parallel, opposite sides equal, four right angles, opposite angles equal, and so forth. When they were asked to shorten their list as much as possible and still give their friend sufficient information to pick out all the shapes of a given type, they would repeat the same redundant list while trying to use fewer English words! Some students gave a list of properties and then said, "But that's a bad definition. It's not the book's definition." We asked one student what the book's definition was, and he said, "Those are the things in red." The book definitions were significant but not appreciated.

Class inclusions were seldom recognized without a lot of probing by the interviewer, even by many students who had had a year of geometry. At the end of the activity, we asked questions about class inclusions. At that point some geometry students went back and relabeled all the figures. For example, in figure 1 they called shape 2 a rectangle as well as a square and shape 9 a parallelogram as well as a rectangle. However, most students balked at the suggestion that these figures had more than one name, even though the shapes fit the "definitions" that they had just given us. Students appeared to be parroting a list of properties they had learned in geometry but were unable to apply the list to a figure. For instance, several students agreed that shape 12 had opposite sides that were parallel, just like their own definition, but that it still wasn't a parallelogram because it didn't look like one. If conflict occurred between the visual and analytical levels of reasoning (levels 0 and 1), the visual level usually won.

*Sorting.* The responses to the sorting activities were consistent with the responses to the drawing activity (see fig. 4). Students who had a year of geometry sorted the figures by type or by some common property. They generated a wide variety of well-defined sorting principles, such as rectangles, at least one right angle, parallelograms, a pair of opposite sides parallel, and perpendicular diagonals. The pregeometry and postgeometry students included sorts that were not mathematically well-defined, such as "these have a big angle" or "these have a long side." The younger students once again relied on predominantly visual sorts, such as "these are diamonds," "these have 'straight' corners," "these are fat," or "these are skinny."

*"What's my shape?"* The list of clues for one of the shapes is given in table 1. The clues were uncovered one at a time, and students were given a chance to think or draw. Then they either asked for another

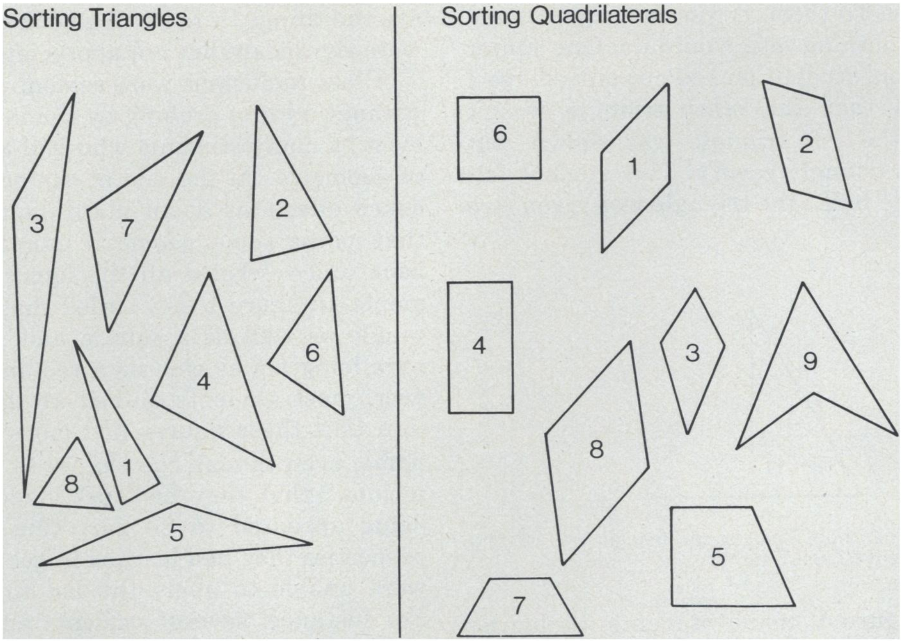


Fig. 4

clue or guessed the shape. Many students in grades K–8 stopped at clue 2 on this shape and declared it to be a rectangle. They were very confident of their answers and felt that exposure to another clue wouldn't change their minds. It didn't, for most of them didn't know a right angle (see clue 3).

Most of the secondary school students used these clues as *necessary* conditions to support a hypothesized shape that they had in mind and then were upset when a clue "torpedoed" their initial guess. For example, the students frequently felt that the shape was a rectangle at clue 2, and subsequent clues acted only as confirmatory evidence for their guess, right up to clue 5. When they saw clue 6 they would sigh, "Oh no!" This type of strategy was used by students whether or not they had taken a geometry course. It was rare that a student used a "cast out" strategy in which quadrilaterals are eliminated until the information is *sufficient* to determine a particular shape. Most of the secondary school students, including those in geometry, were not confident of their solutions and often declared that additional clues might change their minds. Since the clues were used as necessary conditions for a shape rather than as a set of sufficient conditions to determine a shape, students made lots of requests for redundant information. The students seldom arrived at the correct shape with the minimal number of clues. The "litany" of properties observed in the *defining* activity resurfaced. The students frequently made unwarranted assumptions that were not given in the clues; for example, at clue 2 they implicitly assumed that the two long sides were equal in length, opposite one another, or both.

*Axioms, theorems, and proof.* This activity was done only with the secondary school students. Students who had had a geometry course said they had heard of the words *axiom, postulate,* and *theorem,* but many of them confused the terms: "One of them you prove and one of them you assume. I think a theorem is assumed and you have to prove an axiom. Or maybe it is the other way around." Few students could give adequate examples of any of these terms.

We asked the question about quadrilaterals, "Suppose you have a quadrilateral with opposite sides parallel. Must the opposite sides be congruent? How do you know for sure?" Several classes of responses were given to this question: visual, statistical (check lots of cases), deferral to a higher authority, or a prompted proof.

Most students simply tried to draw pictures of quadrilaterals that had opposite sides parallel but not congruent. After several attempts, they concluded that the opposite sides had to be congruent because "I can't draw them any other way." It was quite common for students who had a year of geometry to use this visual (level 0) approach. Some students approached the problem as an exercise in gathering data. "To know for sure, you would have to try lots and lots of cases, and even then you couldn't be completely sure." If a student recognized the need for a proof, she or he might refuse to try to construct one and defer to a higher authority: "There's probably a proof of this in the book" or "I'd ask my teacher to know for sure." With one exception, the secondary school students were unable to construct a proof of this statement or of its converse. With a little help from well-chosen probes, one student did prove the statement. It wasn't until we interviewed a mathematics major, a junior in college, that we found a student who recognized the need for a proof and produced one. In fact, he formulated and logically tested many conjectures throughout his interview, unlike the secondary school students. Of all the students we interviewed, he alone preferred level-3 reasoning.

### Results and Implications for Teaching Geometry

Our investigation of students' conceptions and reasoning processes in geometry was motivated by the van Hiele model of levels of geometric thinking. The van Hieles believed that it was possible to change students' geometric conceptions through teaching activities developed to move a student from one level of thinking to the next. Thus, two main questions arose for us. First, are the van Hiele levels useful in describing students' reasoning processes on geometric tasks? Second, do the levels suggest any changes that seem appropriate for the existing school geometry curriculum? Here is what we found.

1. The van Hiele levels 0, 1, and 2 are very useful in describing students' reasoning processes in geometry, at least on the activities in our interview. Many instances of each of these three types of thinking—visual, analytical, and informal deduction—were uncovered in the interviews.

2. We found no secondary school students who were reasoning at van Hiele level 3. This doesn't mean that such students don't exist, but we didn't encounter any. We suspect that such reasoning is rare at this age.

3. Teachers and students often *confront a level* in a geometry class. That is, it is very likely that the teacher and students are reasoning about the same concepts but at different levels. While the teacher is writing a careful definition of a rectangle on the chalkboard (level 2), the students may be thinking about all the properties that the teacher has left out (level 1).

4. Students may have vastly different geometric concepts in mind than we think they do when we are teaching a course in geometry. Even a concept such as triangle may have many different meanings for the students in a class. When we say and think "triangle," some students include more shapes than we do, and some students limit the use of the word *triangle* to a severely restricted set of figures.

5. The properties of figures are most often viewed as "necessary" conditions rather than as sufficient conditions to determine a shape. Thus the role of definitions is misunderstood, and the need for, and usefulness of, logical inference may not be appreciated.

6. A year after taking geometry, students may regress to a lower van Hiele level. The responses of students after a course in geometry were often analytical (level 1) explorations into components of

figures rather than attempts at informal deduction (level 2), which were more characteristic of the students who were currently taking geometry. In fact, the responses of many postgeometry students were quite similar to those of pregeometry students, except that the former had a better geometric vocabulary. This finding might explain part of the difficulty that beginning college students have with proofs.

7. Two other concurrent projects also investigated the van Hiele levels in different ways. A project in Illinois tested over 2000 high school students and found that over 70 percent were at levels 0 or 1 prior to taking geometry (Usiskin 1982) but that only those students who entered at level 2 (informal deduction) had a good chance of understanding and producing proofs by the end of the course.

Another project at Brooklyn College examined high school geometry textbooks for the van Hiele levels reflected in the textual discourse and problem sets (Geddes et al. 1982). Most of the texts that were studied presented material at van Hiele level 3 and had problem sets that often jumped from level 0 to level 3. That is, the problems were often entirely visual or entirely proof oriented and may have included no questions of an analytical or informal nature (levels 1 and 2).

## Should We Change the Way We Teach Geometry?

Our response would have to be a resounding "yes!" What should we do?

1. *Teach informal geometry to all secondary school students.* A student's introduction to geometry should be informal, without formal proofs or axiomatics, for at least one-half year. Informal activities in geometry should include the investigation of patterns, visualization, properties of polyhedra, similarity, measurement, constructions, transformations (via motions), tessellations, and symmetry. Activities that encourage inference and deduction should also be included, but the writing of carefully structured formal proofs should be omitted. Most students should continue this informal approach for a full year. (A list of references on informal geometry for teachers—materials and places to start—is given at the end of this article.) A traditional formal approach may be appropriate for some students for the remaining one-half year, but even for these students, many topics can be omitted. Usiskin (1980) gives suggestions for eliminating unnecessary "fixtures" in the geometry curriculum.

Many states are currently increasing the amount of mathematics that is required for graduation. At the same time, colleges and universities are stiffening their entrance requirements in mathematics. It would be a mistake to force all students to "take more of the same old thing," especially to force all of them to take a formal geometry course. Our interviews suggest that such a step would be a disaster for both students and teachers.

2. *Develop activities that will move students through the levels.* Most current geometry courses do not contain materials or problems that are designed to help students move from level 0 to level 1 or from level 1 to level 2. The van Hieles found that it was necessary to conduct visual and exploratory lessons with their students to prepare them for deduction. Dina van Hiele developed an entire course based on this premise (van Hiele-Geldof 1957). In an article in this journal, Hoffer (1981) presented an array of prototypical investigations that can help to move a student from one level of thinking to another.

3. *Introduce substantially more geometry in mathematics classes at the elementary and junior high school levels.* It is no wonder that students in high school geometry are unable to cope with inferences about shapes. Many students have had only brief encounters with geometric concepts during their elementary school years. Our investigation suggests that we must allow students to explore geometric concepts and shapes informally for many years prior to a high school course in geometry if they are to develop spatial and visual abilities. Thus, we must provide experiences in informal geometry for students *throughout* their

school years. If the students are thinking at a visual level (level 0), then that level is the one at which we must first address them, regardless of their age.

The mathematics program in the Soviet Union has been greatly influenced by the van Hiele levels, and the study of geometric shapes and solids has been given a high priority in the Soviet pre-secondary school curriculum (Wirszup 1976). In grades 1–3, Soviet children study the properties of geometric shapes, the relations among the shapes, and the measurement of geometric magnitudes. Thus, by the time they enter fourth grade, Soviet children have completed activities that correspond to level 1 in the van Hiele model; they then begin a semideductive study of geometry that continues for the next *seven* years!

## Summary

Interviews were carried out with elementary and secondary school students to examine their levels of thinking about geometric concepts. The results of these interviews corroborate those of other recent studies (e.g., Usiskin 1982), suggesting that although the traditional deductive geometry taught in most secondary schools in the United States is pitched at van Hiele level 3 (deduction), most students are reasoning at level 1 (analysis). As a result, students do not see the need for axiomatics or proof. They acquire poor attitudes toward geometry. We must change the geometry curriculum so that it contains level-1 and level-2 work for our students before we attempt to teach formal axiomatics. Thus, we recommend that

- all secondary school students be taught a course in informal geometry and that
- immediate steps be taken to introduce continual activities involving informal geometry into elementary and junior high school mathematics classes.

Geometry must be given serious attention throughout school mathematics. Reasoning about geometric shapes is a *basic skill* just as much as arithmetic computation (Sherard 1981), and the study of geometric concepts has as high a payoff for problem solving as does arithmetic computation. We should develop informal geometry *parallel* to the development of number concepts throughout a student's school years, not much later as we do now!

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