Project #1 (High School Geometry)

Mathematics 308—Modern Geometry

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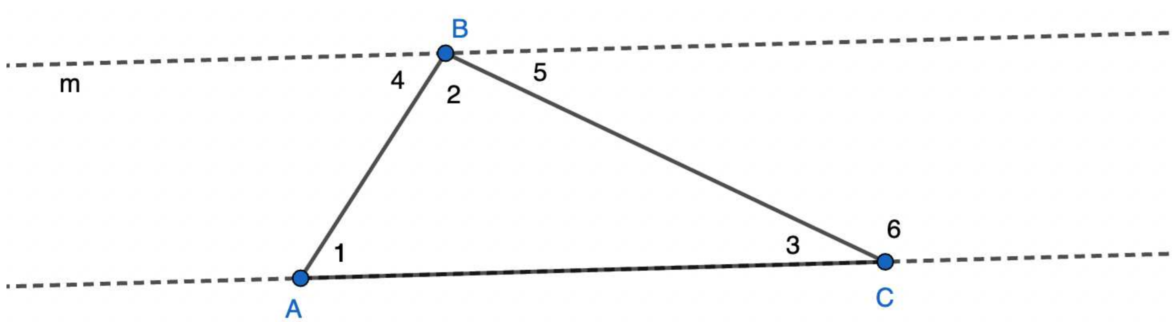
Directions: What follows is a brief review of the major concepts and theorems of elementary Euclidean geometry. Read through the material and then prove the following statements, honoring the given restrictions, using a two-column proof.

1. **Using only the Common Notions, Definitions, Postulates, and T25, prove T05.**

Statement T05: The sum of the angles in a triangle is 180°.

Proof: Refer to the figure below.

Statement	Justification
0. Let $\triangle ABC$ be given	Given
1. Construct line m be parallel to $\overline{AC}$	P5 (Parallel Postulate)
2. $\angle 1 \cong \angle 4$	T25 (Alternate interior angles are congruent)
3. $\angle 3 \cong \angle 5$	T25 (Alternate interior angles are congruent)
4. $m\angle 1 = m\angle 4$ and $m\angle 3 = m\angle 5$	CN3 (Congruence and Equality)
5. $m\angle 4 + m\angle 2 + m\angle 5 = 180°$	Supplementary angles sum to 180°
6. $m\angle 1 + m\angle 2 + m\angle 3 = 180°$	CN4 (Substitution)

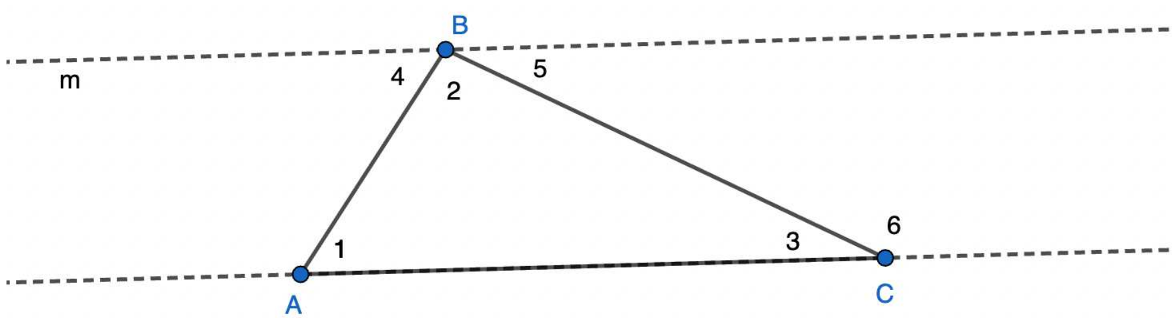


2. **Using only the Common Notions, Definitions, Postulates, and T25, prove T06.**

Statement T06: The measure of an exterior angle of a triangle equals the sum of the measures of the two nonadjacent interior angles.

Proof: Refer to the figure below.

Statement	Justification
0. Let $\triangle ABC$ be given	Given
1. Construct line $\overleftrightarrow{AC}$	P1 (Construct a straight line from any point to any point)
2. $m\angle 3 + m\angle 6 =$ ☐	☐
3. $m\angle 1 + m\angle 2 + m\angle 3 =$ ☐	☐
4. $m\angle 3 + m\angle 6 = m\angle 1 + m\angle 2 + m\angle 3$	☐
5. $m\angle 6 = m\angle 1 + m\angle 2$	☐
6. Similar reasoning establishes the result for the other two exterior angles	Nothing in the argument depends on any particular property of $\angle 6$ other than that it is an exterior angle

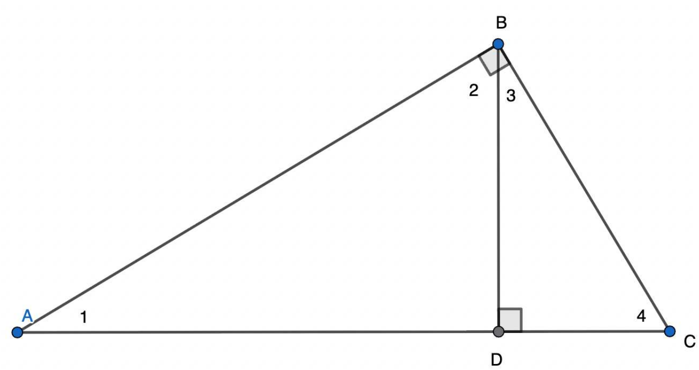


3. **Using only the Common Notions, Definitions, Postulates, and T25 and T07, prove T10.**

Statement T10: The altitude to the hypotenuse of a right triangle forms two right triangles similar to each other and to the original triangle.

Proof: Refer to the figure below.

Statement	Justification
0. Let right triangle $\triangle ABC$ be given, with right angle at vertex B	Given
1. Let $\overline{BD}$ be an altitude of $\triangle ABC$	Given
2. $\angle 1 \cong \angle 1$	
3.	P4 (Right angles congruent)
4.	T07 (AA)
5. $\angle 4 \cong \angle 4$	
6. $\triangle ABC \sim \triangle BDC$	
7.	T08 (Triangles similar to a common triangle are similar to one another)

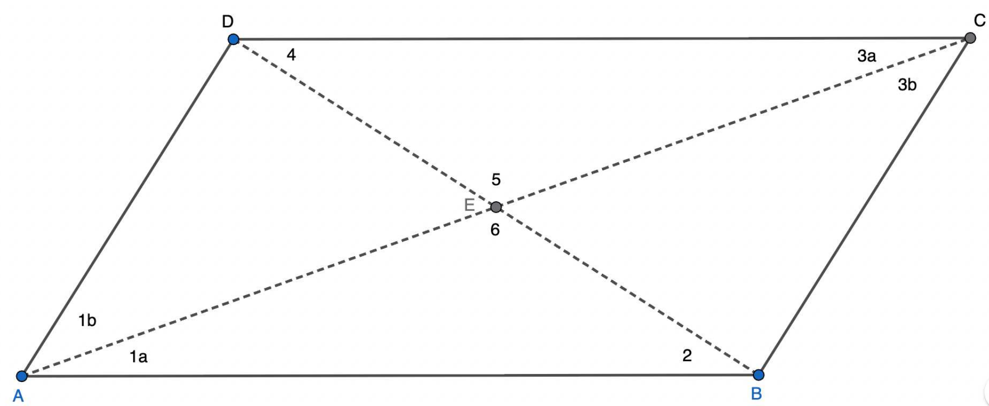


4. **Using only the Common Notions, Definitions, Postulates, and those theorems relating to parallel lines and triangles, prove T11.**

Statement T11: The two diagonals of a parallelogram bisect each other.

Proof: Refer to the figure below.

Statement	Justification
0. Let parallelogram $ABCD$ be given	Given
1. $\overline{AB} \parallel \overline{DC}$	
2. $\angle 1a \cong \angle 3a$ and $\angle 1b \cong \angle 3b$	
3. $\triangle ACD \cong \triangle CAB$	
4. $\overline{AB} \cong \overline{CD}$	
5. $\angle 1a \cong \angle 3a$ and $\angle 2 \cong \angle 4$	
6.	T01 (ASA)
7. $\overline{DE} \cong \overline{BE}$ and $\overline{CE} \cong \overline{AE}$	
8. $m\overline{DE} = m\overline{BE}$ and $m\overline{CE} = m\overline{AE}$	
9. Diagonals $\overline{AC}$ and $\overline{BD}$ bisect one another	

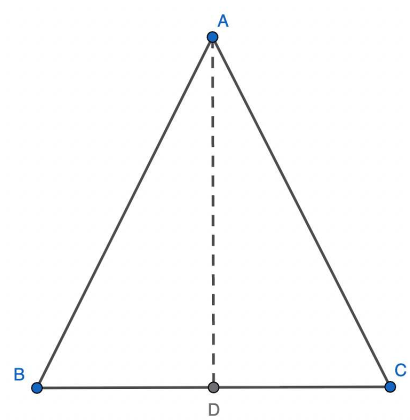


5. **Using only the Common Notions, Definitions, Postulates, and those theorems relating to triangle congruence, prove T12.**

Statement T12: The base angles of an isosceles triangle are congruent.

Proof: Refer to the figure below.

Statement	Justification
0. Let isosceles triangle $\triangle ABC$ be given, with $\overline{AB} \cong \overline{AC}$	Given
1. Construct [ ]	Definition of median and P1 (Constructing a line from any point to any point)
2. [ ]	Definition of midpoint
3. [ ]	CN2 (Reflexivity)
4. [ ]	T02 (SSS)
5. $\angle ABD \cong \angle ACD$	[ ]



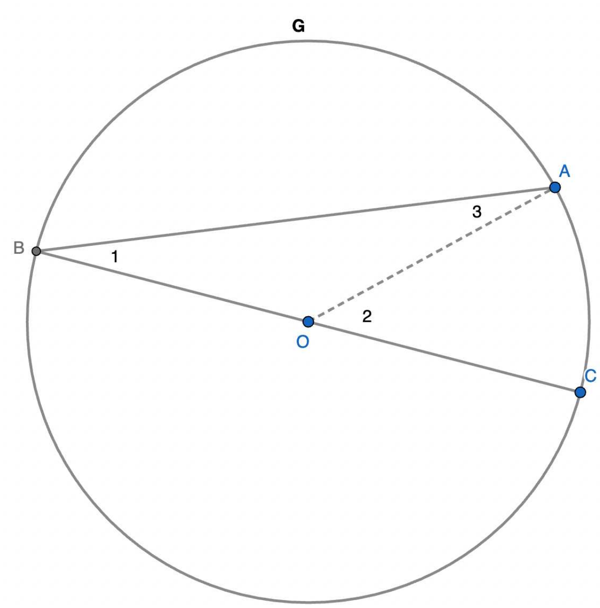
6. **Using only the Common Notions, Definitions, Postulates, and those theorems relating to parallel lines and triangles, along with T21, prove T22.**

Statement T22: An inscribed angle is measured by half its intercepting arc.

Proof:

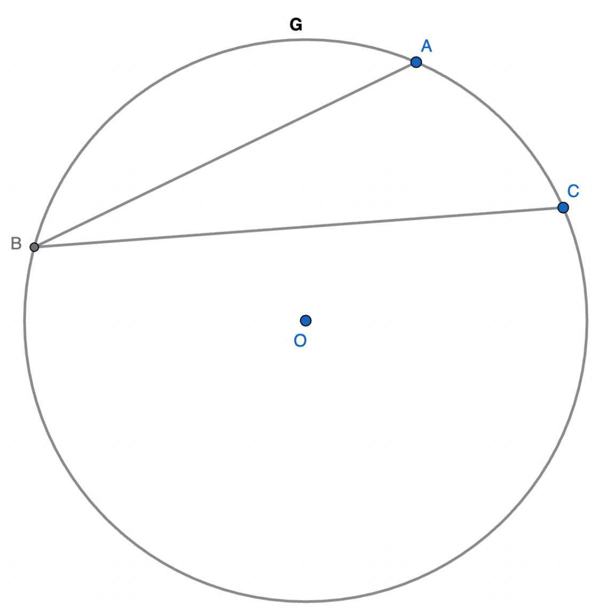
Part 1: Consider first the case in which one side of the inscribed angle is a diameter of the circle. Refer to the figure below.

Statement	Justification
0. Let $\angle ABC$ be inscribed in circle G with center O, and call this angle $\angle 1$, and let side $\overline{BC}$ pass through O	Given
1. Construct segment $\overline{OA}$ and label $\angle AOC$ as $\angle 2$	P1 & P2
2. $\overline{OA} \cong \overline{OB}$	
3.	Definition of isosceles
4.	T12 (Base angles of isosceles triangle are congruent)
5.	CN3 (Congruence and Equality)
6. $m\angle 1 + m\angle 3 = m\angle 2$	
7. $m\angle 1 + m\angle 1 = m\angle 2$	
8.	Basic algebra
9. $m\angle 1 = \frac{1}{2}\widehat{AC}$	



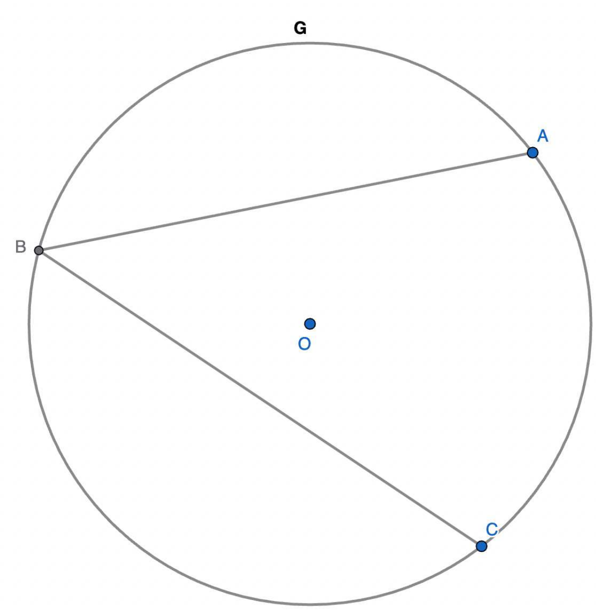
Part 2: Consider first the case in which both sides of the inscribed angle lie on one side of the diameter of the circle. Refer to the figure below.

Statement	Justification



Part 3: Consider first the case in which one side of the inscribed angle lies on one side of the diameter of the circle and the other side of the angle lies on the opposite side of the diameter. Refer to the figure below.

Statement	Justification



Selected Concepts from Elementary Euclidean Geometry*

1. **Common Notions:**

   CN1: (Transitivity) Things that are equal to the same thing are equal to one another.

   CN2: (Reflexivity) Things are congruent to themselves.

   CN3: (Congruence and Equality) Things that coincide with one another, i.e. things that are congruent, have equal measures.

   CN4: (Substitution) If equals be substituted for equals in an equation, the result is an equation.

   CN5: (Adding Equals to Equals) If equals be added to equals, the wholes are equal.

   CN6: (Subtracting Equals from Equals) If equals be subtracted from equals, the remainders are equal.

   CN7: (Whole>Part) The whole is greater than the part.

2. **Postulates:**

   P1: A line can be constructed from any point to any point.

   P2: A finite straight line can be produced continuously in a straight line.

   P3: A circle can be constructed with any segment as its radius and any point as its center.

   P4: All right angles are congruent to one another and therefore have measure 90°.

   P5: Given a line and a point not on the line, a line parallel to the given line and passing through the given point can be constructed.

3. **Definitions:**

   - A segment $\overline{AB}$ is bisected by a point C lying between A and B iff $m\overline{AC} = m\overline{CB}$.
   - If point C bisects $\overline{AB}$, then C is called the midpoint of $\overline{AB}$.
   - An angle is the set of points on two rays with a common endpoint.
   - An acute angle has a measure between 0° and 90°, and an obtuse angle has a measure between 90° and 180°.
   - Two angles are complementary if the sum of the measures is 90°. Two angles are supplementary if the sum of the measures is 180°.
   - Three noncollinear points determine a triangle consisting of the three segments whose endpoints are the given points. Some types of triangles are:
     - *Equilateral:* All sides are congruent.
     - *Equiangular:* All angles are congruent.
     - *Isosecles:* At least two sides are congruent.
     - *Scalene:* No two sides are congruent.
     - *Acute:* All angles measure less than 90°.
     - *Right:* One angle measures 90°.
     - *Obtuse:* One angle is obtuse.
   - An altitude of a triangle is a perpendicular line constructed from one vertex to its opposite side (or to the line containing its opposite side).
   - A median of a triangle is a line constructed from one vertex to the midpoint of its opposite side.
   - The hypotenuse of a right triangle is the side opposite the right angle.
   - A circle consists of all points in a plane equidistant from a given point.
   - A polygon is a plane figure formed by the union of a finite number of linevsegments. The segments meet only at endpoints such that any two segments meet, atvmost at one point, and each segment meets exactly two other segments. Convex polygons have all interior angles less than 180°. Polygons are named according to the number of line segments composing the figures, as follows: Triangle (3), Quadrilateral (4), Pentagon (5), Hexagon (6), Heptagon (7), Octagon (8), Nonagon (9), Decagon (10).
   - Some types of quadrilaterals are:

*Adapted from *Modern Geometries, 5th Edition* by James R. Smart, 1998, Brooks/Cole Publishing Company, Pacific Grove, CA, pp. 409-411.

  - *Square:* Four sides congruent, with four right angles.
  - *Rectangle:* Four right angles
  - *Rhombus:* Four equal sides.
  - *Parallelogram:* Opposite sides parallel.
  - *Trapezoid:* Exactly one pair of parallel sides.
- Polyhedra are three-dimensional figures whose faces are polygonal regions (polygons and their interiors). Prisms are polyhedra with two congruent opposite faces in parallel planes and corresponding edges connected with parallelograms. Antiprisms have two congruent opposite faces in parallel planes, but the edges of one face are connected to the vertices of the other, forming triangular regions for the other faces. Pyramids are polyhedra with a polygonal region for a base and triangular regions for the other faces. A pyramid is formed by connecting a point, not in the plane of the base, with each vertex of the base.
- A circular cylinder is a three-dimensional figure formed by two identical circles in parallel planes, along with the surface formed by line segments connecting corresponding points on the circles.
- A circular cone is a three-dimensional figure with a circular region for the base, along with the surface formed by joining line segments from every point on the circle to a common point not in the same plane.
- A sphere is the set of all points in space the same distance from a given point.
- Skew lines are two lines in space that are neither parallel nor intersecting.

4. **Selected Theorems:**

T00: (SAS) If two triangles have two sides and the included angle of one congruent, respectively, to two sides and the included angle of the other, then the two triangles are congruent.

T01: (ASA) If two triangles have two angles and the included side of one congruent, respectively, to two angles and the included side of the other, then the two triangles are congruent.

T02: (SSS) If two triangles have the three sides of one congruent respectively, to the three sides of the other, then the two triangles are congruent.

T03: (CPCTC) Corresponding parts of congruent triangles are congruent.

T04: The segment joining the midpoints of two sides of a triangle is parallel to the third side, and its length is equal to one-half the length of the third side.

T05: The sum of the measures of the angles of a triangle is $180°$.

T06: The measure of an exterior angle of a triangle equals the sum of the measures of the two nonadjacent interior angles.

T07: (AA) If two triangles have two angles of one congruent, respectively, to two angles of the other, then the triangles are similar.

T08: (Transitivity) Two triangles similar to a common triangle are similar to one another.

T09: If two triangles have their corresponding sides proportional, then they are similar.

T10: The altitude to the hypotenuse of a right triangle forms two right triangles similar to each other and to the original triangle.

T11: The two diagonals of a parallelogram bisect each other.

T12: The base angles of an isosceles triangle are congruent.

T13: The opposite angles of a parallelogram are congruent.

T14: The two diagonals of a rhombus are perpendicular.

T15: The sum of the measures of the interior angles of a polygon of n sides is $(n-2)\cdot 180°$.

T16: If two polygons are similar, then any two corresponding sides have the same ratio.

T17: In a circle or in congruent circles, equal chords are equidistant from the center.

T18: If a quadrilateral is inscribed in a circle, then the opposite angles are supplementary.

T19: The line joining the centers of two intersecting circles is the perpendicular bisector of the common chord.

T20: An angle inscribed in a semicircle is a right angle.

T21: A central angle in a circle is measured by its intercepted arc.

T22: An inscribed angle is measured by half its intercepted arc.

T23: An angle formed by a tangent and a chord to a circle is measured by half its intercepted arc.

T24: The ratio of the measure of a sector of a circle to the measure of area of the entire circle is equal to the ratio of the degrees in the angle of the sector to 360°.

T25: If two parallel lines are cut by a transversal, then the alternate interior angles are congruent and the corresponding angles are congruent.

T26: (The Pythagorean Theorem) For a right triangle, if a and b are the lengths of the sides of the right angle and c is the length of the opposite side (the hypotenuse), then $c^2 = a^2 + b^2$.

T27: The median of a trapezoid has a length equal to half the sum of the lengths of the bases.

T28: The sum of the measures of the vertex angles in a convex polygon of n sides is $(n-2) \cdot 180°$. If the polygon is regular (all sides congruent and all angles congruent), then the measure of each vertex angle is $\dfrac{(n-2) \cdot 180°}{n}$.

T29: The volume of any prism/cylinder is the (area of the base)×(height). If the shape is a pyramid, then the above formula is multiplied by $\frac{1}{3}$.