Project #2 – Rigor

Mathematics Education 322—Modern Geometry
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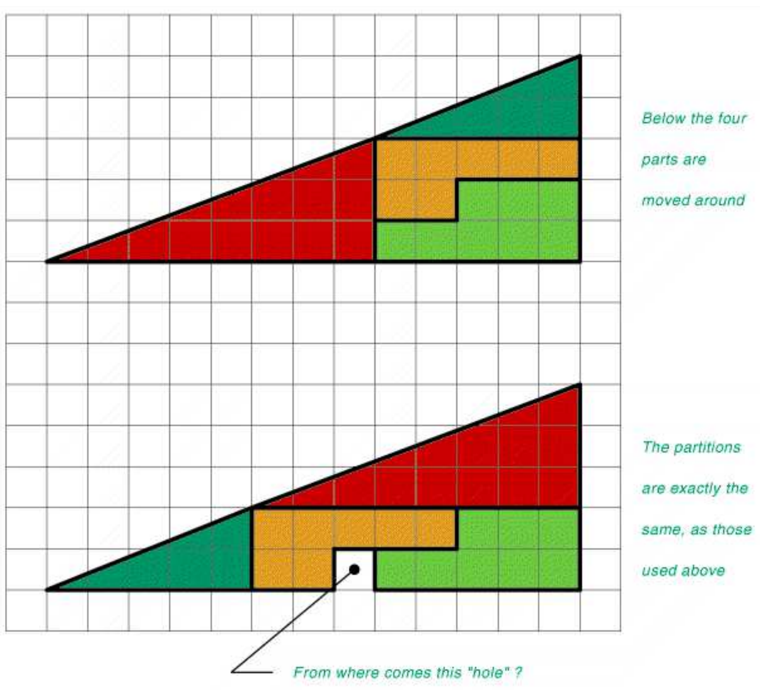
1 Puzzles

1. **Missing Area:** Watch the *Puzzle Rearrangement* video linked on the course web page. Another version of this puzzle is given below. Solve it.



2. **Infinite Gold:** Determine the flaw in the "construction of infinite gold" shown in class.

3. **Betsy Ross Construction:** Show how Betsy Ross' construction of a 5-pointed star "with one snip" works; i.e. use computations to show that the folds will result in, say, a regular pentagon at some point or some other definitive geometric shape that will result in a regular 5-pointed star.

4. **Proof by Induction:** This very powerful method of proof is essentially a formal method of saying, "et cetra." For example, suppose we wish to prove that a statement is true for all integers n. To do this, we show that it is true for $n = 1$, and then make an argument that it must also be true for $n = 2$, and then another argument that it is true for $n = 3$, etc. We begin to notice that each argument looks the same: we refer to the previous case and argue from it that the next case works. After a few iterations of this, we could just say, "etc." and let the reader notice that the pattern of our thinking always works, but we formalize this process in the following manner:

$$(\text{True for } n = 1) + (\text{If true for } n, \text{ true for next case}) \longrightarrow (\text{True for all } n.)$$

Now, consider the proof of the following theorem:

Theorem: All M&M's are the same color.

Proof: We use induction on the number of M&M's. If $n = 1$, then clearly there is only one M&M, and it is the same color as itself. Suppose that the result holds for $n = k$; we will show that it holds for the next case as well; i.e. the case where $n = k + 1$. Consider any group of $k + 1$ M&M's. First order the M&M's from left to right. Consider the group, F, of the first k M&M's. Since there are only k M&M's, they are all of the same color. Call that color C. Now, consider the group, L, of the last k M&M's. Since it also contains only k M&M's, all of the M&M's in this group are also the same color. However, since the second M&M appears in groups F and L both, and the color of the second M&M is C, then all M&M's in group F must also be color C. Hence, all $k+1$ M&M's are the same color, C. Thus, we have shown that if the result holds for $n = k$, it must hold for $n = k + 1$. Ω

Most people would claim that their senses tell them that not all M&M's are the same color. So, was Plato correct? Are your senses deceiving you? Is pure rational thought the only way to uncover this sacred truth that all M&M's are indeed the same color? Is there something wrong with the Method of Proof by Induction? (Admit it, you always thought the method was kind of hokey.) Explain precisely how to reconcile the proof above with the experience you have gained through your sense of sight.

5. **Einstein's Special Theory of Relativity and Deductive Reasoning:** Einstein reasoned, some years ago, that the speed of light is a constant as measured from anyone's point of view, traveling at any speed, in the universe. The following quote from Nigel Calder's *Einstein's Universe* will help you to appreciate how bizarre this conclusion is:

> "That the speed of light in empty space remains constant in most circumstances is one of the trickiest aspects of relativity to understand. It was Einstein's prime assumption and it has been fully justified by the evidence. Nevertheless it puzzled Einstein's contemporaries, and with good reason. In the case of sound waves the speed depends, just as you might expect, on how you are moving through the air that carries the sound.* No such variations occur in the case of light... Let a shaft of sunlight in through a laboratory window on the Earth and measure its speed. 186,000 miles a second. Now, fly in a spaceship towards the Sun at half the speed of light. Again, it is 186,000 miles a second. Turn around and fly away from the Sun at half the speed of light. The sunlight overtakes you at a stubborn 186,000 miles a second."†

Nevertheless, it appears to be true. As Calder says, "[With this assumption] an effect upon time is deducible at once, taking us to the heart of Special Relativity."‡ Indeed, with a little help from elementary geometry, you will derive Einstein's central result for his Theory of Special Relativity.

Consider the following situation, which Einstein would call a "thought experiment." You and a friend leave Earth in two separate spaceships at the same time, traveling at a constant velocity, v, so that you remain parallel to each other, d meters apart, throughout the trip. Dr. Peratt remains behind on Earth to watch you. Now, realize that once you leave the earth, there is no fixed frame of reference by which you can judge your concludes that you are not moving at all, but are rather in perfect rest. You feel no effects of motion, since your velocity is perfectly constant. You now decide to measure the speed of light by firing a laser beam of light to your friend's spaceship and measuring the length of time, t, it takes for the beam to reach your friend's ship and return to you.

(a) Draw a diagram which shows the path that the light beam will take from the travelers' point of view. Draw another diagram which represents the path the light beam will take from stationary observer's point of view. Remembering that velocity=distance/time, express the speed of light, c, in terms of d and t.

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*There appears to be a contradiction between Calder's statement, "the speed of sound depends, as you might expect, on how you are moving through the air that carries the sound," and the truth that, no matter how fast you travel through the air, the sound you generate travels at the same speed, since it is a wave traveling through a medium and hence has a speed affected only by the medium. In fact, there is no contradiction, which serves to highlight the absolutely radical nature of Einstein's claim. If I am traveling at $50\frac{m}{s}$ and the speed of sound in the medium at which I am traveling is $350\frac{m}{s}$, then the sound which I give off does not travel at $350\frac{m}{s} + 50\frac{m}{s} = 400\frac{m}{s}$. Rather, it travels at $350\frac{m}{s}$, since its velocity depends only on the medium in which it travels. However, if I am traveling at $50\frac{m}{s}$ toward a sound wave which is traveling at $350\frac{m}{s}$, then the wave, though traveling at $350\frac{m}{s}$ relative to the air, will appear to me to be traveling at $400\frac{m}{s}$. This usually intuitively makes sense to most folks. The nature of light, however, is completely different. What Einstein is saying is that in this last scenario, with light, the answer would be quite different: if I am traveling at half the speed of light towards a light wave, then, unlike the sound wave, the apparent speed of that light wave will be, not 1.5 times the speed of light, but simply the speed of light. This results from the fact that light doesn't travel in a "medium." Even though light behaves very much like a wave, it is in fact very much a particle as well and can travel even in empty space. Sound, on the other hand, has to travel in a medium since it has no mass or substance to it—it is not something that can be touched; rather, it is a disturbance in the medium (e.g. air). One of the surprising outcomes of Relativity was that light, even though it behaves like a wave, actually has mass and substance. Einstein called this mass a "photon." Because light has this property, it can dislodge electrons when it strikes an atom, and we currently use this property to make solar panels which convert solar energy into electricity. Yet, light demonstrates properties that only waves demonstrate (interference for one) and in fact follows Maxwell's equations for electromagnetic *waves*. That light can be both a particle and a wave is one of the many paradoxes arising from Relativity. It's great stuff if you want to mess with your head.

†Calder, Nigel. *Einstein's Universe*, 1986, Penguin Books, New York, NY, pp. 173-174.

‡Calder, Nigel. *Einstein's Universe*, 1986, Penguin Books, New York, NY, p. 152.

(b) Now, suppose that Dr. Peratt, who is observing all of this, also wants to calculate the speed of that light beam. He will also use the formula velocity=distance/time to do so, but his calculation will differ, as explained in class.

(c) Use the two calculations of the speed of light to relate the two values t =the number of seconds that pass from the travelers' point of view and T =the number of seconds that pass from the stationary observer's point of view. You should derive a formula of the form $T = f(t)$, where $f(t)$ involves only c, v, and t, where c is the speed of light in a vacuum (approximately 186,282 miles per second).

(d) According to this formula, if you were to travel in your spaceship at a velocity of $180,000$ miles per second for one day, how much older would a stationary observer be than when you left Earth?

(e) How fast would you have to travel in order for you to be gone for one week according to your reckoning, but have one millennium pass on Earth? Express your answer also as a percentage of the speed of light.

(f) In light of inductive and deductive reasoning, explain Calder's judicious choice of words when he says, "It was Einstein's prime assumption and it has been fully justified by the evidence," and, again, "[With this assumption] an effect upon time is deducible at once. . . ." Concentrate on the use of the words "assumption," "justified," and "deducible."

(g) Now, consider Einstein's statement: "If the speed of light is constant for every observer, then time must retard at high velocities." Suppose that we now uncover evidence that, in fact, the speed of light is not constant for every observer. What can be said regarding the truth value of Einstein's statement? Explain. Suppose that we uncover evidence that time doesn't always retard at high velocities. What can be said regarding the truth value of Einstein's statement? Explain.

2 Axiomatic Systems

1. **Axiomatic Systems:** Read Section 1.5 of your text, and then consider the following axiomatic system.

(a) For each of the four theorems, determine whether it is true or false. If it is true, prove it. If it is false, provide a counterexample. In your proofs, you should resist the tendency to "create" an entire model of the axiomatic system and then point to that model as "proof" that the theorem is true. The reason this is not valid is that your model may not be the only possible one that satisfies all of the axioms, and therefore other properties that it possesses may not necessarily follow from the axioms but instead from the particular choices that you made in the construction of your model. One famous example of this is our current model of the real numbers as an uncountable set. As it turns out, there is a way to satisfy all of the axioms of the real numbers with a model that is countable, but no one uses that model. Hence, most people assume that the reals are uncountable because they need to be, not because we chose them to be that way. Therefore, when you create a model, all that shows is that there is *at least one* way to fulfill all of the axioms at the same time.

So, attempt to use only those axioms needed to prove the specific theorem. That is, aim for a "minimal proof" that provides the shortest logical route from the axioms to each theorem. Many times for these proofs, that will involve a proof by contradiction.

(b) Determine whether or not the system is consistent, independent, and complete.

Axiomatic System:

Axiom 1: Every cause involves at least one radical.

Axiom 2: Every cause involves at most two radicals.

Axiom 3: $\exists$ at least three radicals.

Axiom 4: For any pair of radicals, $\exists$ one and only one cause involving both.

Axiom 5: Each radical is involved in exactly two causes.

Theorems:

Theorem 1: $\exists$ exactly three radicals.

Theorem 2: Each cause involves exactly two radicals.

2. **Peratt Geometry:** Consider the following axiomatic system for a geometry called Peratt Geometry:

Primitive (Undefined) Term: point.

Definition: a *line* is a set of points.

Axiomatic System:

1. Every line contains at least three points.
2. Every two points, A and B, lie on a unique line.
3. If lines AB and CD intersect, then so do lines AC and BD (where it is assumed that A and D are distinct from B and C).
4. There exist at least two distinct lines. (Alternatively, we could have said, "There exist at least 4 points, no three of which lie on the same line.")

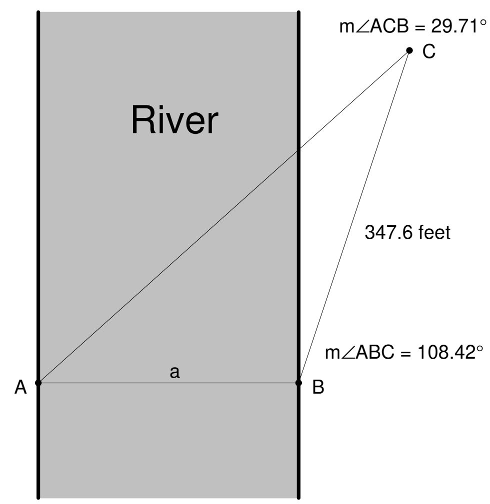
(a) Build a model for this system to show that it is consistent relative to Euclidean geometry.

(b) Find the smallest possible such geometry; i.e. find the smallest number of points and lines which fulfill all of the axioms.

(c) Repeat the above exercise if Axiom 4 is eliminated.

3. **Non-right Triangle Trigonometry:**

(a) *Civil Engineering:* Suppose that a surveyor wishes to measure the width of a river at a certain point along the river. She uses a theodolite to take measurements as shown in the figure below. Use her measurements to find the width of the river.



(b) *Surveying:* British surveyors in the middle 1800's were sent to India to survey the country, particularly Mt. Everest, culminating in "The Great Trigonometrical Survey of India." As they traveled toward Mt. Everest, they stopped periodically to take measurements with a transit theodolite (see the figure below).[§] Suppose that they stopped at one point and measured the angle of elevation to be 3.144°. After traveling 75 miles more toward the mountain, they again take a reading and find the angle of elevation to be 12.391°. From this information, they determine the height of Mt. Everest, as must you to complete this problem. Express your answers in feet.

(c) *Geoscience:* Layers of rock often recede into the ground at an angle, as seen in the figure below. The angle between the layer of rock and the horizontal ground level is called its "dip." Often, what is exposed on the surface gives a misleading view of the thickness of the rock strata. We will examine such a case in this problem.

[§]1841: Sir George Everest, Surveyor General of India from 1830 to 1843, records the location of Everest; 1848: Peak b is surveyed the British, which ruled India; The height is calculated at 30,200 feet from measurements taken 110 miles away; 1852: The Great Trigonometrical Survey of India determines the Peak XV is the highest mountain in the world; 1854: Peak b renamed Peak XV; 1856: Surveyor Andrew Waugh completes the first height measurement, declaring Everest to be 8840 meters high (29,002 feet); 1865: Peak XV re-named Mt. Everest to honor Sir George Everest, the Surveyor General of India. Everest called Chomolungma in Tibet and Sagarmatha in Nepal; 1999: National Geographic Society finds that the summit of Mt. Everest is 6 feet higher than previously thought, adjusting the height from the accepted value of 29,029 feet to 29,035 feet.

In the following figure, there is a layer of rock that is buried underground. The only exposed surface is along the slope of ground in the front of the figure; the layer of rock appears to be 18 feet thick. But, this is not the true thickness of the layer of rock, as you can see from the figure below. Some measurements are taken. We find that $\measuredangle ECF = 220°$ and that plane $ABCE$ is horizontal. Using a drill, we measure a depth of 8 feet at D before hitting the layer of rock and a depth of 11.5 feet at C, which is 12 feet from D.

- Find the true dip of the layer (i.e. find $\measuredangle GHI$).
- Find the true width of the layer (i.e. find the length of w).