Project #3

Mathematics 308—Modern Geometry

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1 Impossibility Proofs

1. *Extension Fields:*

   (a) Show that the following numbers are algebraic by finding a polynomial with integer coefficients of which the given number is a root.

   - $3 + \sqrt{7}$
   - $\dfrac{3 + \sqrt{7}}{5}$
   - $\dfrac{2 + 5\sqrt{3}}{3 - 2\sqrt{7}}$
   - $\sqrt[3]{2 + \sqrt{11}} + 5$

   (b) Prove that if $a + b\sqrt{n}$, where a, b, n are in some field $\mathbb{F}$, is a root of $8x^3 - 6x - 1$, then $a - b\sqrt{n}$ must also be a root of $8x^3 - 6x - 1$.

2. *Impossibility Proofs:*

   (a) Prove that it is impossible to construct a square with the same area as a given circle using only Euclidean Tools.

   (b) In our proof of one of the lemmas relating to the trisection of an arbitrary angle, we stated, "since the roots must sum to zero...". Prove that, in fact, the roots of an arbitrary cubic, $ax^3 + bx^2 + cx + d$ must sum to $-\frac{b}{a}$. (I would prefer that you do this on your own, without Googling the answer. Thus, I will give you a hint. If $r_1,\ r_2$, and r_3 are the roots of the given cubic, then we know that the cubic can be written as $a(x - r_1)(x - r_2)(x - r_3)$ by the Fundamental Theorem of Algebra. Take it from there, and make sure to explain how we know that the a (and not some other constant) must be out front of the factored form.)

2 Constructing Loci

1. *Parabola:* One definition of a parabola is "the locus of points equidistant from a given point, called the *focus*, and a given line, called the *directrix*." Use Geogebra, with the *trace* feature, to construct a parabola from this definition. The point that I move in order to trace out the parabola should be labeled as such. Export your construction as parabola.ggb.

2. *Ellipse:* One definition of ellipse is "the locus of points such that the sum of the distances from the point to two given points, called *foci*, is constant." Use Geogebra, with the *trace* feature, to construct an ellipse from this definition. On your sketch should be a line segment with a point on it that I can slide to change the nature of the ellipse. The point that I move in order to trace out the ellipse should be labeled as such. Export your construction as ellipse.ggb

3. *Hyperbola:* One definition of hyperbola is "the locus of points such that the difference of the distances from the point to two given points, called *foci*, is constant." Use GSP, with the *trace* feature, to construct a hyperbola from this definition. Export your construction as hyperbola.ggb.

4. *Mystery Locus:* Use Geogebra to create the locus of points that are equidistant from a given point, called the *focus*, and a given circle, called the *directrix*. Describe the loci that you obtain when the focus is within the circle and prove that it is what it appears to be. Repeat with the case when the focus lies outside the circle. Export your construction as circleloci.ggb.

5. *Another Mystery Locus:* In the previous construction, trace the midpoint between the focus and the center of the circle, and prove that the resulting shape is what it appears to be.

3 Miscellaneous Problems

1. *Connecting Conic Sections to Analytic Geometry:* Consider the parabola $f(x) = ax^2$. Prove that a line perpendicular to the directrix of this parabola and passing through a point $(p, f(p))$ on the parabola makes the same angle with the parabola at that point as does the line segment from $(p, f(p))$ to the focus of the parabola. (It may eventually be helpful to employ the fact that the cosine of the angle between two vectors is given by $\frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}$. If you choose not to use this method, you can employ right triangle trigonometry to get the same result, albeit with a bit more work.)

2. *Constructions using "Data":* A *datum* is a set of n elements, any $n-1$ of which determine the remaining one. Each of the following sets are a datum of which $n-1$ elements are given. You are to construct the remaining item. Please construct these by hand.

   (a) Construct a triangle, given the measure of one angle, the length of an adjacent side, and the length of the altitude to that side.

   (b) Construct a triangle, given the length of one side, the length of the median to that side, and the circumcenter.

   (c) Construct a circle with a given radius and tangent to two given intersecting lines.

   (d) Construct a circle with a given radius tangent to a given line and tangent to a given circle.

3. *Angle Trisection:* Prove that the Tomahawk Method of angle trisection presented in class does, in fact, work.

4. *The Golden Ratio:* Find the Golden Ratio in the construction of a regular pentagon shown in class.

5. *Hexaflexagons:* Watch the videos on Hexeflexagons on the course web page and make a well-labeled Hexaflexagon and Hexahexaflexagon.

6. *Self-Working Card Trick:* View the self-working card trick video on the web page and then explain, using abstract thinking (i.e. labeling unknowns using variable, etc.), explain why it must work they way it does. Also, identify any portions of the trick that are unnecessary and which are just added for psychological effect.

4 Triangle Constructions

Famous Nine-Point Circle: There are many theorems regarding the coincidence of certain lines constructed from a triangle. For example, one can prove that for any triangle, the perpendicular bisectors of each side all meet at a single point, called the *circumcenter* (which is, in fact, the center of the circle that circumscribes the triangle). In fact, the medians, interior angle bisectors, exterior angle bisectors (of two angles and the interior bisector of the third), and altitudes of a triangle meet in points called the *centroid, incenter, excenter,* and *orthocenter*, respectively.

The famous "Nine-Point Circle" is the circle defined by the three points where the altitudes of the triangle meet their base. In 1765, Euler showed that this circle also passes through the midpoints of the sides of the triangle. By Feuerbach's theorem, the nine-point circle also passes through the midpoints of the segments that join the vertices and the orthocenter. Create a dynamic construction which demonstrates these properties of the Nine-Point Circle (see Figure 3 below), and export it as "nine.ggb." Be sure to drag around points on the triangle to ensure that the nine-point circle remains. Also, label the midpoints of each side of the triangle with a label that moves along with the point as you change the triangle.

The Nine Point Circle