Test #1, Part #2—Solutions

Mathematics 308—Modern Geometry
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Directions: On a separate sheet of paper, neatly answer the following questions.

1. **van Hiele Levels:**

   (a) State each van Hiele level and briefly (in one sentence) describe its main characteristics.

   Answer: See the handout for answers. You needed to do more than list the level and give a vague description of it for full credit. You needed to identify *the* main aspect of that level that distinguishes it from all of the others.

   (b) Give a concrete example of reasoning at van Hiele level 1 (assuming they are numbered 0 through 4).

   Answer: See the handout for possible examples.

2. **Axiomatic Systems:** The three important properties of an axiomatic system are consistency, independence, and completeness. For each of these terms, define it and explain briefly how one can establish that a particular axiomatic system possess it.

   Answer:

   (a) *Consistency:* means that none of the axioms inherently contradict the others. This is established by building a model of the axiomatic system – that is an object that satisfies all of the axioms; if the axioms were inherently contradictory, such a model would be impossible to build.

   (b) *Independence:* means that none of the axioms can be proven from the others. This is established by replacing the axiom in question with its negation, and then building a model of the resulting axiomatic system. If in fact the axiom in question could be proven from the others, then replacing it by its negation would result in an axiomatic system in which the original axiom were true, being proven from the others that remain, and false, since its negation would be present in the new axiomatic system. In this were the case, no model would be able to be constructed of the new axiomatic system.

   (c) *Completeness:* means that there are no statements that can be formulated within the axiomatic system that cannot be either proven or disproven. Kurt Gödel proved that almost every axiomatic system is necessarily incomplete.

3. **Short Answers:**

   (a) Aristotelian Logic is characterized by what "law" that distinguishes it from other types of logic, such as Fuzzy Logic and Eastern Mystical thought?

   Answer: Law of the Excluded Middle.

   (b) Cite the main argument against heliocentrism that was advanced by Aristotle and which Galileo could not counter (and in fact was not definitively countered until 1838).

   Answer: Absence of parallax in viewing the stars during different seasons of the year.[1]

   (c) Briefly describe the difference between a corollary and a lemma.

   Answer: A corollary is a trivial consequence of a proven theorem, whereas a lemma is a lesser proven result that is used to prove a theorem.

   (d) Briefly describe the great insight among the Greeks concerning abstract axiomatic systems identified by Hvidsten (the author of your text).

   Answer: For full credit, you had to identify the idea that Greeks classified objects, thereby enabling themselves to prove results about entire classes of objects and not just individual objects. This is what allowed them to progress beyond the Egyptians in their mathematical understanding.

[1] Aristotle reasoned that parallax error should cause the position of stars to look relatively different when the earth is on one side of the sun vs. the opposite side of the sun. In fact this parallax error is present, but was too small to be discerned until telescopes were sufficiently advanced, in 1838.

(e) Identify the two logical traps that are inherent in the development of any axiomatic system and the way in which the Greeks overcame them.

Answer: One logical trap is circular reasoning, and the Greeks overcame this trap by the introduction of unproven statements called axioms. The other trap is a never-ending stream of prior statements, and the Greeks overcame this trap by the introduction of undefined terms.

(f) Epicurus maintained that proving some of the more elementary results of Euclid was a waste of time, since those results are obvious. He famously quipped that "even a donkey knows to take the hypotenuse of a triangle to get to a stack of hay." Proclus responded by noting that proving such "obvious" truths serves two purposes. List them.

Answer:

(1) what is obvious to our senses is not always true, and is certainly not sufficient for scientific inquiry.

(2) By carefully proving elementary results, we train our minds so that, in those situations where our intuition and perception do fail us, we can still reason carefully and accurately.

4. **Short Paragraph Answers with Choice:**

(a) Answer *one* (1) of the following:

- Describe the main contribution of Bertrand Russell (1872-1970) to set theory. Be specific. Your explanation should include the phrases "set of all sets that are not elements of themselves" and "self-referential."

  Answer: Bertrand Russell illustrated the need for a more precise treatment of set theory. Until his time, a set was defined to be "a collection of objects." Russell demonstrated that this definition leads to nonsense by defining a set P to be the set of all sets that are not elements of themselves. He then asked the question, "Is $P \in P$ or not?" If $P \in P$, then P is one of those sets that are not elements of themselves, so $P \notin P$. On the other hand, if $P \notin P$, then P is *not* one of those sets that are not elements of themselves, so P must be an element of itself; i.e. $P \in P$. In summary, if $P \in P$, then $P \notin P$ and if $P \notin P$, then $P \in P$. This is nonsense.

- Describe the main contribution of Kurt Gödel (1902-1978) to modern mathematics. Be specific. Your explanation should include the phrases "if P is consistent, then…" and "if P is not consistent, then…."

  Answer: Kurt Gödel proved, precisely, two things:

  – Given a consistent axiomatic system that contains the natural number system, the consistency of such a system cannot be proved from within the system.

  – Given a consistent axiomatic system that contains the natural number system, the system must contain undecidable statements, that is, statements about the terms of the system that are neither provable nor disprovable.

(b) Answer *one* (1) of the following:

- George Sim Johnston observed that "it was not simply a new theory of the nature of celestial movements that was feared, but a new theory of the nature of theory." Explain briefly what he meant by that. You explanation should include the phrase "saving the appearances."

  Answer: Until the dawn of empiricism, theories were looked upon merely as a convenient way of organizing data ("saving the accidents (appearances)"). What was being advanced with Galileo is that, if a theory "saves all of the accidents" (i.e. perfectly organizes all of the data), then it is actually identical with truth.

- Timothy O'Neill identified the Renaissance a "curiously conservative and rather retrograde movement in many ways." Explain briefly what he meant by that, and how it relates to Galileo's main criticism of his contemporary scientists that caused him to label them "little Aristotles."

  Answer: The ancient Greeks advanced a rich tradition of critical thought, inquiry, and experimentation. The Renaissance movement venerated the knowledge of the Greeks but did not necessarily embrace their spirit of quest for knowledge, preferring instead to uphold the results of great thinkers like Aristotle as "untouchable." It caused them even to disregard truly venerable advances in knowledge from medieval thinkers in favor of holding in high esteem only the newly rediscovered knowledge of the ancient Greeks. This is why Galileo labeled his detractors as "little Aristotles," because they refused to enter into the spirit of inquiry formulated by the ancient Greek thinkers and therefore wouldn't progress beyond what Aristotle had discovered, especially if it meant contradicting some of Aristotle's conclusions.